A Geometric Alternative to the Riemann Zeta Function

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Overview:

This new alternative to the Riemann zeta function is a finite sum of real vectors that finds all the Riemann zeros. When displayed graphically the new function, identified here as kappa, is found to have beautiful symmetry and many intriguing characteristics. This paper demonstrates some of those features. It is hoped the kappa function may lead to insights concerning the zeta function and Riemann hypothesis.

Kappa, the New Function:

$$\kappa(a,b)=\sum_{n=1}^{b/\pi}\left(\frac{1}{n^a}\right)[(\cos(b\ln(n)) \quad , \quad \sin(b\ln(n)))] \tag{1}$$

Three aspects of this function should be noted immediately: first, the summation is over a finite number of terms; second, all values are real numbers; and third, it is a summation of vectors. That is, the argument of the summation is a scaler times the address of a vector in Cartesian two-space, not in the complex plane.

Contrasting the zeta and kappa functions:

1. zeta has an infinite series of terms; kappa has a finite number of terms
2. zeta uses complex numbers in the complex plane; kappa is real in the Cartesian plane
3. zeta requires analytic continuation to expand its domain; kappa does not
4. zeta may be as precise as is desired in its determination of zero values; kappa perhaps not

Skipping ahead of explanations, Image 1 shows a plot of the kappa function for which the last spiral is coincident with the origin for particular ‘a’ and ‘b’ values: a Riemann zeta zero. Image 3 shows ‘a’ and ‘b’ values for which the last spiral is *not* at the origin: not a Riemann zeta zero.

Early Discoveries:

The most surprising discovery is that this new kappa function indicates Riemann zeros with a finite number of terms. Of course, ‘n’ must be an integer to sum n terms, and ‘b’ and π are certainly not. So the limit on the sigma notation of b/π is really b/π rounded to an integer. Another early surprise was the incredible symmetries displayed in plots of the function. The kappa function can best be understood via plots of its features, many of which are collected in this paper. One of the zeta zeros first used was the zero at b = 60000.441207279. This turned out to be very fortunate since that zero best displays the behavior of the kappa function at a zero as well as the astounding symmetries and beauty of the function. Image 1 shows that early image of the kappa function at a=0.5 and b=60000.441207279. This particular set of ‘a’ and ‘b’ values is used so frequently that I refer to it as the ‘60k’, the ‘k’ meaning times 1000.

Image 2 shows several of the most important features of the kappa function plot. When values for the variables ‘a’ and ‘b’ of a known Riemann zeta zero are chosen the function’s sum of vectors returns to the origin with the last spiral. Also, when corresponding vectors and spirals are connected with Correspondence Lines (blue), the Symmetry Line (green) perpendicularly bisects the correspondence lines. The correspondence lines shown in the images are spiral inflections drawn to vector addresses. The medial inflections also form parallel lines perpendicular to the symmetry line. These medial correspondence lines are drawn from the medial inflections to the midpoints of the opposite vectors. In every graphic there is a feature I refer to as the saddle (red) that appears at term number $n = \sqrt{(b/2\pi)}$. The saddle always straddles the symmetry line.

Image 3 is the same graphic as Image 2 except ‘b’ has been changed to a non-zero value. Notice the last spiral does not center on the origin, which is true of all non-zero ‘b’ values. The correspondence lines are still parallel to each other and are bisected by the perpendicular symmetry line.

Not all kappa function plots appear as simple as is the ‘60k’ plot shown in the preceding figures. In images with ‘b’ values less than a few thousand the above described features are very hard to discern. Across the spectrum of ‘b’ values most images are very knotted up, and although all the features (saddle, symmetry line, etc.) are present they are difficult to separate from the general clutter. The 60k image is not special in any way other than having easy to discern features. Image 4 is of a zero valued image that is rather typical and shows that the symmetries are there, the last spiral is on the origin, but the clarity is diminished.

There are two additional general attributes to examine before exploring the specific details of the kappa function: that of a ‘last spiral’ and the function’s quasi-convergence. All standard kappa function plots regardless of ‘a’ and ‘b’ values start with individual vectors which become more and more tangled until somewhere around term number $n=\sqrt{(b/2\pi)}$, after which the vectors organize into larger and larger spirals centered around n values of $n=b/k\pi$, for k=1, 3, 5, ... Odd k values are where the deflection of one vector from the previous vector passes through an integer factor of the value π. When k=1 the last spiral is formed and its graphical position indicates whether or not the ‘a’ and ‘b’ values are a Riemann zero.

Each spiral consists of an inward-bound clockwise spiral until near the center when it changes to a counterclockwise outward-bound spiral at an ‘inflection point’. There are other inflection points in the image which happen halfway between every pair of spirals. These points are where the deflection of one vector from the previous vector passes through a value of 2π. These inflection points, called medial inflections to differentiate them from the spiral inflection points, happen at term numbers $n=b/k\pi$ for k=2, 4, 6, ... Even k values give medial inflections, odd k values give spiral inflections. Image 5 shows both the spiral and medial points.

There is always a last spiral and a last inflection point, after which the outward-bound spiral continues without limit in a logarithmic spiral. The kappa function, equation (1), is easily seen to have two parts: a positive length, the $1/n^a$ value; and an angle, the b*ln(n), which translates into an (x, y) address: x, the cosine term, and y, the sine term. The argument of the cosine and sine functions is an angle, in radians, and not modulo 2π. The value of ‘b’ is always positive, and since ‘n’ is always positive, the natural log of ‘n’ is always non-negative. For a given ‘a’ and ‘b’ as ‘n’ gets larger the length of each vector becomes smaller, and the raw angle (not mod 2π) becomes larger. When the cosine and sine functions are applied the resultant vector’s angle to the x-axis is limited to an angle of 0 to 2π. But in the case of a spiral image what is important is the raw angle (b*ln(n)) of each vector relative to the raw angle of the preceding vector; that is, the difference in raw angles. Within a spiral this difference in raw angles becomes smaller and smaller and at some point it passes through the value of kπ where k is a positive odd integer that varies inversely with ‘n’. This is the spiral inflection point. Then the spiral starts its outward journey.

For the medial inflections a similar event occurs, except the raw angle difference between two adjacent vectors passes through kπ, where k is a positive even integer. Thus the inflection points are determined by the values of an integer factor of π.

How all this leads to there being a last spiral is best explained by deriving equations for term numbers at certain relative raw angles. The raw angle is given by:

$$raw\,angle = b*\ln(n) \tag{2}$$

and the relative angle between a vector term number ‘n’ and its successor ‘n+1’ is:

$$\begin{gathered} relative\,angle\ =\ b*\ln(n+1) - b*\ln(n) \\ rel.ang. =\ b*(\ln(n+1)/\ln(n)) \\ rel.ang. =\ b*(\ln(1+\frac{1}{n})) \end{gathered} \tag{3}$$

and solving for the value of ‘n’ given a specific relative raw angle is:

$$\begin{gathered} rel.ang.\ =\ b*(\ln(1+\frac{1}{n})) \\ n\ =\ \frac{1}{e^{\frac{rel.ang.}{b}} - 1} \end{gathered} \tag{4}$$

An example of using equation (4) follows:

Given a = 0.5 and b = 60000.441207279 (a Riemann zero) the term number when the relative angle is π is calculated as:

$$n\ =\ \frac{1}{e^{\frac{\pi}{60000.441207279}} - 1}\ =\ 19098.23362 \tag{5}$$

For comparison, b/π for a ‘b’ of 60000.441207279 is 19098.73361. All the inflection points can be calculated this way and each is found to be very close to the value of b/kπ, for k = 1, 2, 3, ... In fact, the second-to-last spiral with raw angle difference of 3π is at (rounded) term number n = 6366, and the intervening medial inflection at a raw angle difference of 2π is at n = 9549. There can be no spirals beyond k = 1. A way of realizing this is to recognize that the relative angle, equation (3), is always positive and approaches zero monotonically. An angle of zero is equivalent to an angle of 2π which would be a medial inflection, but equation (3) never reaches that value. This means that for the kappa function the upper limit on sigma is finite and is b/kπ, k=1, or just b/π.

It is tempting, in looking at the π, 2π, and 3π term numbers above to conclude that the 'n' value for 2π is just 1/2 the π value and that the 3π value is just 1/3 the π value, but while very close these are not correct. Also, in this work inflection points were taken to be rounded values of b/kπ, not the rounded value of the very slightly more correct equation (4).

<u>Graphical Explorations</u>:

Varying 'b':

'b' is the primary tool for finding Riemann zeta zeros. Setting 'a' to 0.5 (which is, of course, assuming the Riemann Hypothesis to be true) and incrementing 'b' in small steps causes the last spiral to roam around the plane until it lands on the origin, indicating a zero. Image 6 shows this phenomena for Riemann zeros at b = 7034.292019875 (green) to b = 7036.042983450 (red). The increment amount of 'b' is 0.1 except for the last increment which was a bit less so as to land on the origin, a zero.

As 'b' is incremented the plot winds up on itself, looping around once for each zero. This is easily seen starting with the '60k' zero and displaying successive zeros. Images 7 through 13 demonstrate this with the first four zeros being successive zeros but then jumping to the 22nd zero after '60k', then the 50th zero, and finally the 51st zero after the '60k'.

(Note: the JavaFX graphics automatically (and without remedy) 'miters' the ends of line segments, which means if two lines intersect with a very small angle the miter will extend well beyond where the lines actually end. This results in some 'spikiness' that is an artifact of the program.)

Another fascinating result found when slowly incrementing the 'b' value is that the points already defined as correspondence points actually become coincidental as the plot winds and unwinds. If we start with the familiar '60k' we see that the last inflection point is coincident with the origin, which we could call point 0, the starting point. The next spiral correspondence is inflection point b/(3π) with point 1, followed by the spiral inflection at b/(5π) and point 2. These coincidences shown in Images 14, 15, 16, and 17, continue to happen up the chain of correspondences.

Not only are the spiral inflection points coincident with the vector points, the medial inflections are coincident with the midpoints of the vectors. This is shown in Images 18, 19, and 20. In fact, if one were to draw straight lines between spiral inflection points those lines and the vectors would scissor together with equal ratios, shown in Image 21.

Varying 'a':

In the kappa function the 'a' parameter is interpreted as determining the length of each vector. Thus each vector's length is $1/n^a$.

Varying the value of 'a' seems to make it obvious that anything other than 0.5 won't work; no zero solutions exist for other than 0.5 values. Obvious or not, while very instructive and perhaps leading to new insights, nothing is proven.

Starting with known zeros, varying 'a' moves the last spiral away from the origin, never to return. Images 22 and 23 show this effect. In black is a zero, Image 22 uses the '60k' plot and Image 23 uses 60002.388346887, a rather tangled image. In red the value of 'a' is increased making the denominator smaller and the whole plot bigger. In green the value of 'a' is decreased producing a smaller plot. Again, an existing zero can only have a=0.5. As for non-zero starting values, Image 24 for b=60001.31759209299 shows results similar to Images 22 and 23. In all three of these images the increment or decrement of the ‘a’ value is 0.02 per step.

An even better illustration of the effect of ‘a’ on the kappa function can be seen by holding ‘b’ constant and varying ‘a’ from 0 to 1. If the last inflection point is plotted for each value of ‘a’ the set of possible values of the kappa function for the given ‘b’ is shown. Image 25 shows the resulting curve for the ‘60k’ value of ‘b’. Image 26 displays that same image plus the next dozen ‘b’ values of known zeros.

For non-zero values Image 27 displays the ‘a’ tracks from 0 to 1 for ‘b’ values ‘60k’ to about 60,001, incrementing ‘b’ by 0.1. Only the first track, ‘60k’, is a zero.

(The Images 25 - 27 actually only show ‘a’ from about 0.3 to 1 as the curve continues off screen.)

These images provoke the most interest in finding a path to the Riemann hypothesis.

Varying N and (1-a)/b:

Capitol ‘N’ is the rounded value found by equation (4) with a relative angle of π, approximated by b/π. The sigma summation has lower bound of 1 and upper, *finite*, bound of N. If N is less that b/π a quasi-convergence cannot be reached. Any term number ‘n’ larger than ‘N’ just adds vectors to the outward-bound last spiral. A low value of ‘b’ at a zero best demonstrates this. Image 28 shows the Riemann zero at b=49.773832478, with N=16 (the b/π value, rounded), in black and going to an n=6000 in red. The outward bound spiral after the *last* inflection point of every plot is a logarithmic spiral. The spiral in Image 28 is very close to the equation (6):

$$r = 0.1 * e^{0.01 * \theta} \tag{6}$$

The spiral outward from the last inflection point of the zero at b=14.134725142 has the equation:

$$r = 1.06 * e^{0.0354 * \theta} \tag{7}$$

From the two examples in equations (6) and (7), and a few others, an empirical conclusion can be drawn. The standard form for a logarithmic spiral is:

$$r = c * e^{d * \theta} \tag{8}$$

where ‘c’ simply re-sizes the self-similar spiral and is used in this instance to start the spiral coincident with an arbitrary spiral of the plot, and ‘d’ determines the rate at which each spiral radius increases. What seems to be the case is the value of ‘d’ is equal to 0.5/b.

Testing many different values of ‘a’ and ‘b’ confirms that the outward-bound last spiral of the kappa function plot is logarithmic and that the *value of the ‘d’ in equation (8) is actually (1-a)/b*.

Testing the spirals within the plot, for example the spiral at b/3π, indicates those spirals are not logarithmic. Those internal spirals may be Euler’s spirals, further work needs to be done to determine if that is the case. It would be no surprise to find Euler’s name already on a feature of the new kappa function.

Inflection Points and Quasi-convergence:

The term ‘inflection point’ is somewhat misleading. As used in this work an inflection occurs when a series of lines (or vectors) are arranged so that, when traversed from one end to the other, the lines start out with each line turning in the same direction from its predecessor, say, to the left, and then suddenly change to every remaining line turning the other way, to the right.

A spiral can either spiral in to a point, spiral out from a point, or spiral in and then also spiral out. This last case can only happen if, at the center of the spiral, there is an inflection. All the spirals in the kappa function are of the in and out type; they all have an inflection at their centers. In the abstract continuous model of mathematical functions built on real numbers and that are everywhere differentiable, the inflection is, in the limit, a point. In the world of discrete numbers, specifically in the case of the kappa function, the inflection occurs at a line (or vector) segment. The end points of this segment are each an inflection point, depending on which spiral one is following inward. Image 29 demonstrates these concepts.

So, in the discrete world, where is the ‘center’ of a spiral? Somewhat arbitrarily the midpoint of the vector whose end points are the two inflection points is chosen as the center of the spiral.

The quasi-convergence of the kappa function to zero (a vector address of (0, 0)), is then defined as: a kappa function zero, for a = 0.5, and as ‘b’ is increased, is the ‘b’ value where the distance from the midpoint of the last vector, after moving closer to the origin, is closest to the origin and then starts moving away. More simply, when the midpoint of the last vector is closest to the origin. If ‘a’ is not 0.5, the last midpoint’s closest approach to the origin does *not* indicate a zero value for that ‘b’ value.

Spirals Close Up:

Most of the images presented so far have been at a screen magnification of 100. The images in this section are all the last spiral of the ‘60k’ plot with N=19098, but at different magnifications. The first of these, Image 30, is magnified 10,000 times. The center is seen as a filled black circle. There is, however, a hole in the center that is obscured by the vectors running across the center of the figure. Backing off the zoom a little and instead of showing vectors showing only the dot of each vector’s address gives Image 31, at a magnification of 3,000. Without the lines running across the center the hole becomes visible. Image 32 is of the hole at a magnification of 100,000. The vectors, if put back in this image, would not form spirals around the edge of the hole, rather, they are going back and forth across the hole forming acute relative angles with each other. When we zoom in further the end points (dots) of the vectors are out of the frame of the image - all we can see are the vectors’ lines. At a magnification of 3,000,000 a beautiful pattern emerges. Image 33 shows a small white gap crossing very near the origin. It is bordered by the two last vectors in the ‘60k’ plot. The last one, on the left, is the b/π vector. The next vector would be the start of the last outward-bound logarithmic spiral. The midpoint of the last vector is about $2x10^{-7}$ units away from the origin, and the vector length itself is about 0.007 units long.

From this analysis we can see that if one of the end points of the last vector were used as the center it would be 0.0035 from the origin, while its midpoint is around 10,000 times a better measure.

The Saddle:

Around a ‘b’ value of 5000 the saddle becomes quite apparent. It appears in every image regardless of ‘b’ value but is so degraded at lower values as to be unrecognizable, and is often hidden in the tangle of vectors in many images with larger ‘b’ values. The saddle was at first a wholly visual attribute, easily seen in some plots but of no notable importance. In fact, it along with most other aspects of the plots, can be best seen in a single loop kappa function plot, like the ‘60k’. It was apparent that the saddle divided the single-loop plots into a left half made up of spirals and a right half made up of vectors. Discovery of the correspondence lines led at first to drawing the symmetry line running from the origin to somewhere in the middle of the saddle, rather than a perpendicular bisector of the correspondence lines. Further research discovered that, although the saddle is always bisected by the symmetry line, that is not the defining attribute of the symmetry line. The symmetry line is best defined as ‘a line perpendicular to and bisecting all (any) correspondence lines’. The symmetry line then ‘follows’ the curves rotation with changing ‘b’ values, maintaining the symmetry whether or not the ‘b’ value is that of a Riemann zero.

Image 3a shows a non-zero ‘b’ value with the newly defined symmetry line, and a few representative correspondence lines. Compare this image with Image 3, displaying the symmetry line defined as ‘a line from the origin to the saddle’.

All of the spirals alternate with medial inflections. The saddle, even though preceding (in n term number) any spirals, seems to have a spiral at each end, but with two medial inflection points between them. This gives a twist to the direction spirals rotate. In general, spirals rotate clockwise when spiraling inward with increasing ‘n’ and counterclockwise spiraling outward. The spiral preceding the saddle is clockwise inwards and counterclockwise outwards, with increasing ‘n’. But after the saddle this seems to be reversed until the formation of recognizable spirals which revert to the norm. This behavior is not yet understood.

The calculation of each vector’s address (the x and y coordinates) is precise and its plot on the graphics is also precise. The coordinates and the axes are delineated with real numbers. But calculations for finding the center of spirals (the spiral inflection points) are of necessity approximations. The last vector of a zero does not necessarily pass exactly through the origin.

A closer look at the ‘60k’ plot illustrates these considerations. We assume the correspondence lines and the symmetry line are ‘supposed’ to intersect perpendicularly. There is no logical reason to assume this, just intuition from the visual symmetries. We next use a medial correspondence line that goes to a medial vector that is almost parallel to the correspondence line, this greatly limits the error if we are ‘off by one’ vector. Finally, we draw three symmetry lines from the origin to: the truncated value of √(b/2π), that value plus 1, and that value minus 1. The graphic produced is Image 34. A magnified version is given in Image 35. Actually putting a protractor on the image the angles are, to within visual perception, 89º, 90º, and 91º. (All done using a Riemann zero and the original symmetry line.)

The √(b/2π) value for the saddle is empirical; another way to find the saddle is to equate the spiral count ‘k’ with the vector count ‘n’ through the correspondences. The ‘k’ of b/kπ is found to correspond with an ‘n’ of 2n+1. That is, a ‘k’ of 3 (the second to last spiral) corresponds with an ‘n’ of 1 (the first vector). Setting them equal and solving for ‘n’:

$$\frac{b}{(2n+1)*\pi}=n$$
$$n = \sqrt{\frac{b}{2*\pi}+\frac{1}{16}} - \frac{1}{4} \tag{9}$$

which for the ‘60k’ is 97.4693, where √(b/2π) 97.7208.

The saddle has become an important feature of the kappa function plot, but remains unexplained as to its formation and placement within the plot. In fact, all of the empirical discoveries, the b/π last inflection point, the b/kπ medial and spiral inflection points, the √(b/2π) saddle, the symmetry line, and most recently found, the (1-a)/b logarithmic last outward bound spiral, are at this point not fully explained. Just how these simple formulas mathematically arise from the kappa function is yet to be discovered. The math explains the formulation of the inflection points through the relative vector angles, but does not yet explain the massive symmetries observed.

‘n’ Not an Integer:

The kappa function is built around the sigma notation ‘Σ’ that stipulates the addition of an integer number of terms. Interestingly, and what Hofstadter might label a ‘strange loop’, each term to be added in a sigma formula might contain the term number in the calculation. The limits imposed on the sigma symbol are a lower limit, generally something like ‘n=1’, and an upper limit, some positive integer. The increment of ‘n’ for each term is always 1. As far as the kappa function is concerned, the important thing is not that the limits are integers but that they are at uniform intervals. That is, each term number is the previous term number plus a constant value. Programmatically this is implemented as a loop with a starting value, an ending value, and an increment value, and the program doesn’t care if any of the three values are integer or not. A uniform increment is indicative of a self-similar type structure.

The kappa function depends on the value of the term number for two calculations: each vector (that is, each term) has a length of $1/n^a$, and an angle of b*ln(n). It is amazing that as long as the starting value and increment value are the same positive real number the kappa function plot is almost the same. Almost the same, as it will be rotated from the standard plot by an angle of b*ln(start value) and the plot will be larger or smaller than the standard by a factor of √(1/(increment value)). The other difference is that the ‘N’ value, the number of terms needed to reach the last inflection point is now (b/π) times the increment value. All symmetries and features are the same as the standard plot and the kappa function continues to find all the Riemann zeros. There is no apparent increase in precision in using something other than 1 as the increment. The kappa function can now be expressed as:

$$\kappa(a,b)=\sum_{n}^{nb/\pi}\left(\frac{1}{n^{a}}\right)[(\cos(b\ln(n)) \quad , \quad \sin(b\ln(n)))]; \qquad n\in\Re\,, \;\; increment \;\; by \;\; n \tag{10}$$

Equation (10) is a non-standard use of the sigma notation; an upper and lower limit as well as an increment value must be indicated. Image 36 shows several different values used for both the start and increment variables.

There is at least one more anomaly that can be discussed. The main use of the kappa function is to find Riemann zeros. The discovery of its massive symmetries suggests the kappa function may be another path to proving the Riemann hypothesis. But there is a version of the kappa function that is not at all symmetric *but still finds the zeros*. The summation limits for this non-symmetric kappa function are a starting value of any number and an increment of exactly twice that number. Image 37 shows the ‘60k’ plot with starting value 0.5 and increment of 1.

The ‘b’ Track Display:

The ‘b’ track type of image is commonly seen in published descriptions of the Riemann zeta, and the kappa function can produce a similar graph as shown in Image 38. This is a plot of the last spiral inflection point as ‘b’ is increased. The value of ‘b’ is a Riemann zero (assuming a=0.5) whenever the curve passes through the origin. There is a difference, however. Because of the necessity of rounding from real numbers to integers the low numbered ‘b’ values have jumps and backtracks in the curves that are also there in the higher ‘b’ values but are smoothed out by the ratio of the rounding error to the size of the rounded number. Image 39 shows this effect.

Most satisfying, with the kappa function there is something visual to explain the creation of the ‘b’ track plot: the graphical location of the last spiral inflection point as ‘b’ is increased.

The Circle Display:

The standard display adds the vectors in the order of the summation of terms (‘n’ order), while the circle display adds the vectors in the order of their respective raw angles (b*ln(n) order), starting with the least angle greater than 0 and ending with angles of 2π. (In the standard display the first vector is always angle 0, but if placed first in the circle display it lessens the symmetry, so it is placed last at the equivalent angle of 2π.) The circle display is a complete re-ordering of the vector summation resulting in a very different plot. Image 40 shows the ‘60k’ circle display. It is an almost perfect circle.

Magnifying the image 6,000 times we see the last vector doesn’t quite reach the origin, even though this is a Riemann zero. Image 41 shows this difference. That is as expected, in fact the distance from the last vector in the circle to the origin is about 0.0036, which is half the length of the last vector in the standard display. When the last ‘n’ vector’s length is reduced to 1/2 it normal length (in other words, stopping at its midpoint) the error is reduced as in the standard display.

Other Displays:

If all the vectors are shown as originating at the origin the ‘spokes’ plot is created as shown in Image 42. This demonstrates the distribution of angles and lengths.

Another display reveals just the x-coordinate values displayed on the y-axis in term (‘n’) order. That is, the positive y-axis is used for the term numbers and the x-axis is the value of each term’s x-coordinate. Image 43 shows the x values on top of the standard display for the ‘60k’. Both the x- and y-coordinates can be displayed on top of the standard plot as well. Image 44 shows the zero at b=6025.65298906 with both x and y displayed.

The value of ‘n’ plays two roles in the kappa function, and in the zeta and almost all sigma notation equations: that of a counter, and of a numerical value. The kappa function can be considered to have three variables: ‘a’, ‘b’, and ‘n’. This leads one to wonder what a three dimensional plot of these three variables might look like. In programming language such a plot would be called a ‘point cloud’. Programs like Blender allow for creating and displaying point clouds by facilitating zooming, rotating, and panning the image in real time 3D. Capturing ‘stills’ of these images does not begin to allow the full experience of the 3D graphs, but it does indicate the capabilities. Image 45 shows a zero valued kappa function with the coordinates of each vector plotted against the x- and y-axes and the term number ‘n’ on the positive z-axis.

Another type of point cloud maps the ‘a’ value from 0 to 1 on the x-axis, the ‘b’ value, in this case, from 1000 to 1100 on the y-axis, both with increments of 0.001, and the distance the last inflection is from the origin on the positive z-axis. Image 46 shows part of this plot, which, again, loses much of its value as a tool in still image form.

Summary:

Presented here is the kappa function, a new, geometrical way of seeing the Riemann zeta zeros. There are no proofs offered here, only a hope that this path may lead others a little further in discovering the nature of primes. There is much left to explore.

Notes and Images:

The Author:

Robert Kern Duncan, born 1948 Camden, New Jersey, USA.

Education:
Masters at Teaching, Mathematics; 2006
University of South Carolina, Columbia, South Carolina

BS Mathematics; 1975
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BS Computer and Information Science, College of Engineering; 1970
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Experience:

Among my varied careers most have been in the computer industry. My last and most enjoyable endeavor, however, was teaching high school mathematics, computer programming, and computer graphics and videography. During that time I was teaching an advanced math class for college credit in the high school as an adjunct professor to a community college.

Since developing the kappa function almost a decade ago, I have continued to explore its intricacies. It is not the only area I'm working in: positive integer mathematics, positive axes coordinate systems, Gaussian primes, Feigenbaum logistic maps, two and three dimensional fractals, and the theory of mathematical operations are a few of my interests .

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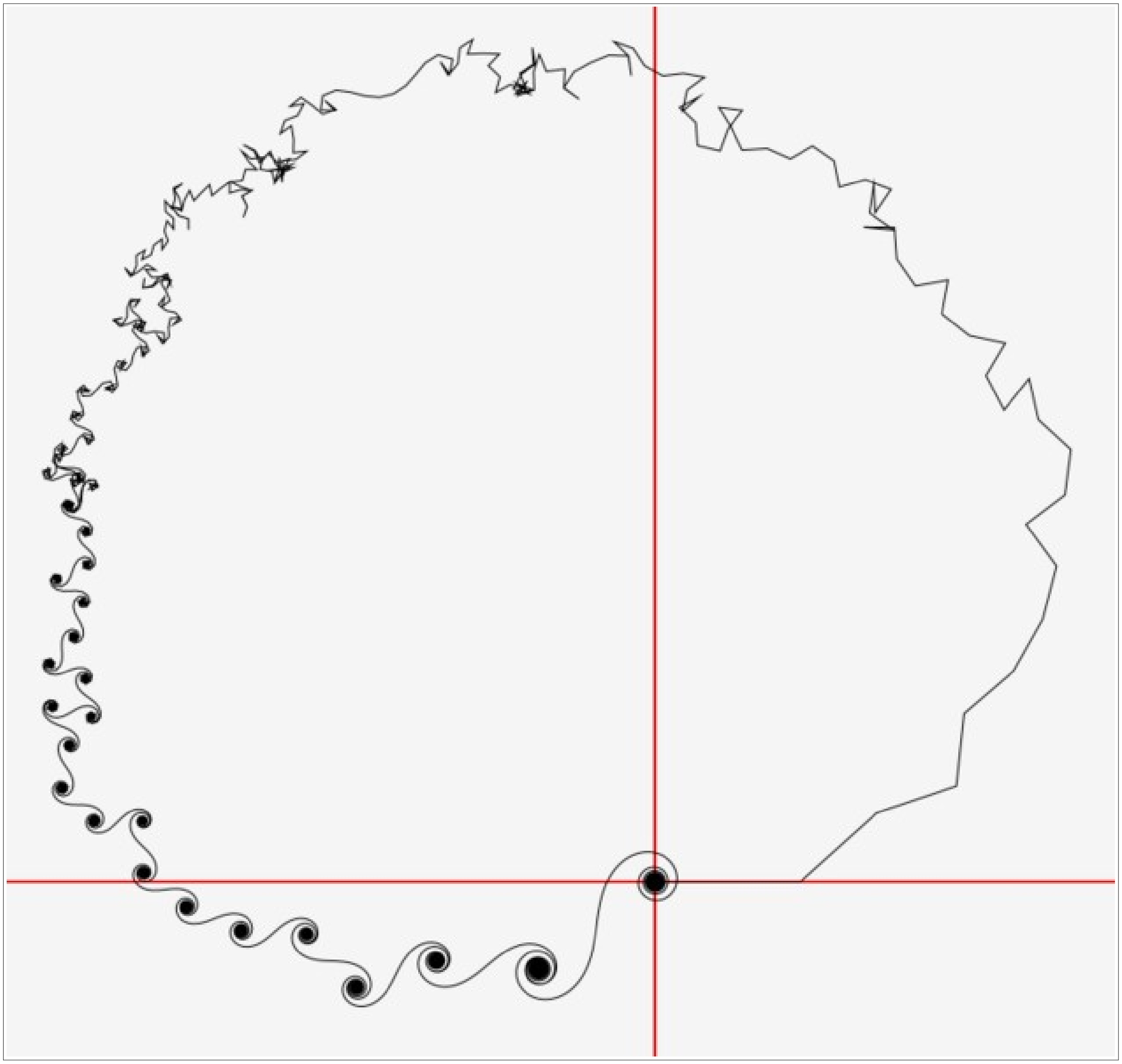
Hardware and Software Tools:

Hardware:
AMD Ryzen 7 5800x 8-core processor x 16
32 GiB memory
6.8 TB disk capacity
Radeon RX550/550 Series graphics card

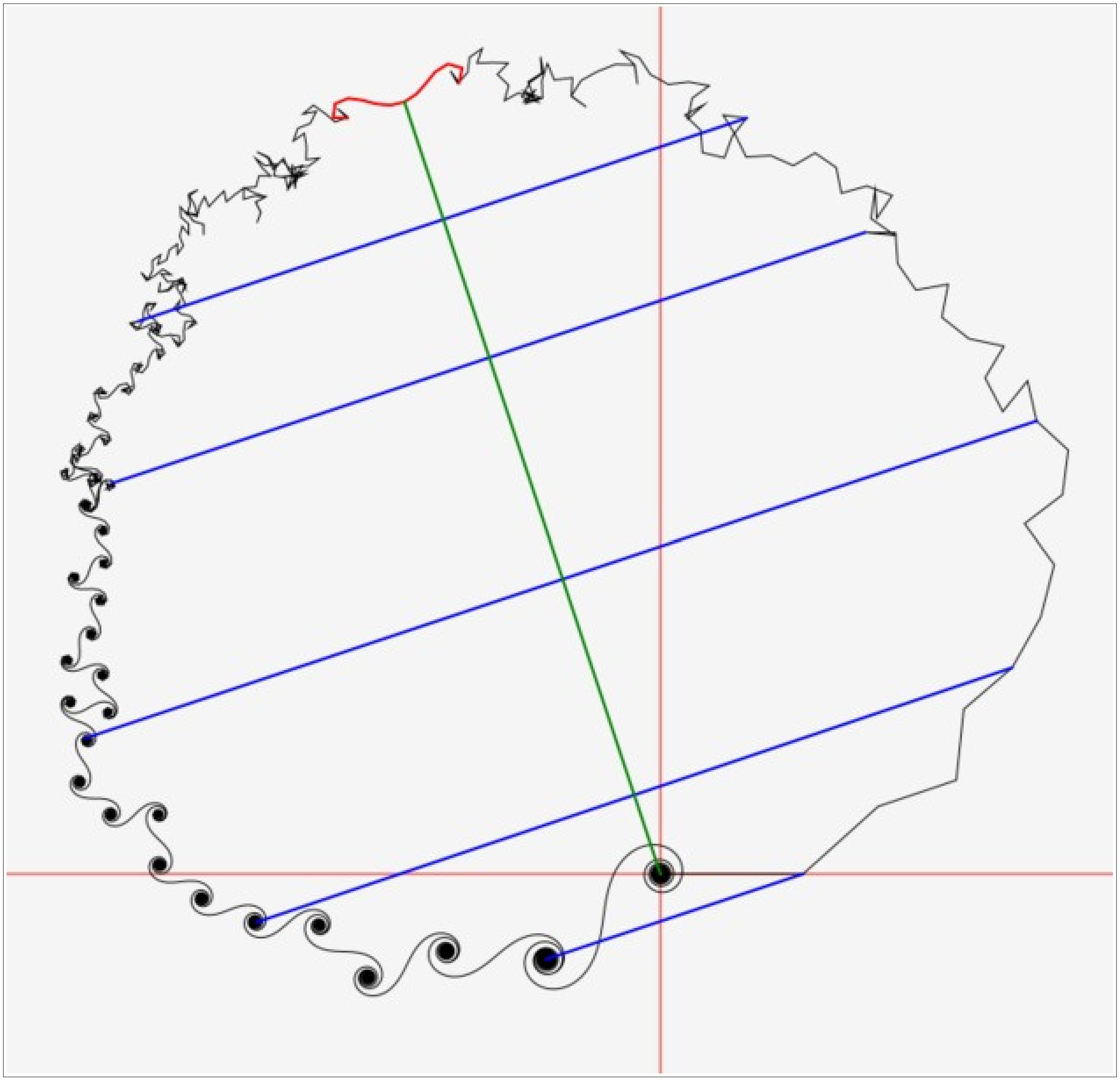
Software:
Debian 11
JavaFX 21
IDEA IntelliJ 2022.3.2
Blender 4.2.5
LibreOffice 7.0.4.2
Screenshot
Linux mogrify

Image 1



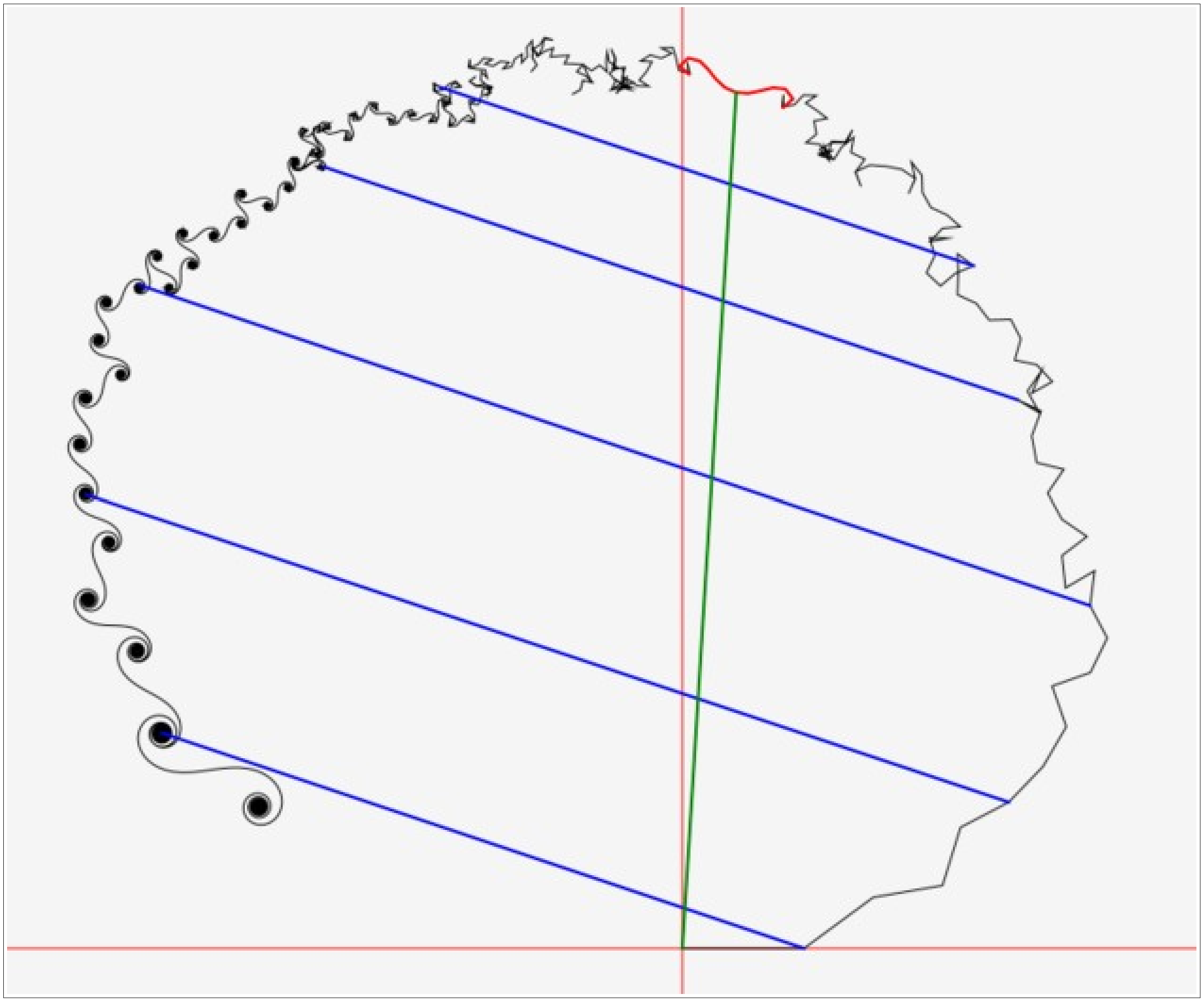
One of the earliest images of the kappa function, the ‘60k’. Parameter values for this image are: a = 0.5, b = 60000.441207279, and the number of terms (vectors) is N = 19098. This is a Riemann zeta zero.

Image 2



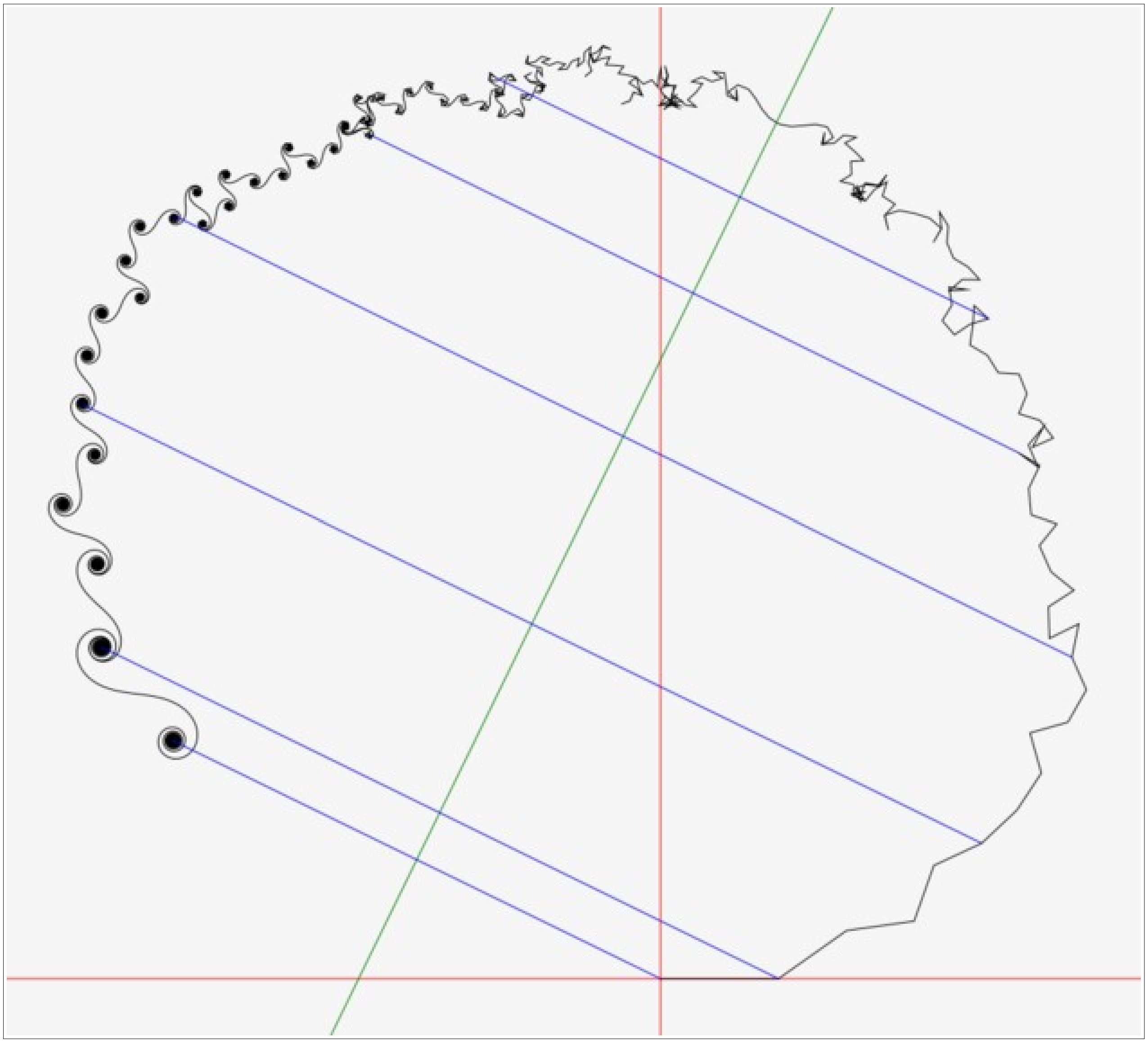
The '60k' with labels. The saddle in red, the symmetry line green, the correspondence lines in blue, and the X and Y axes in light red.

Image 3



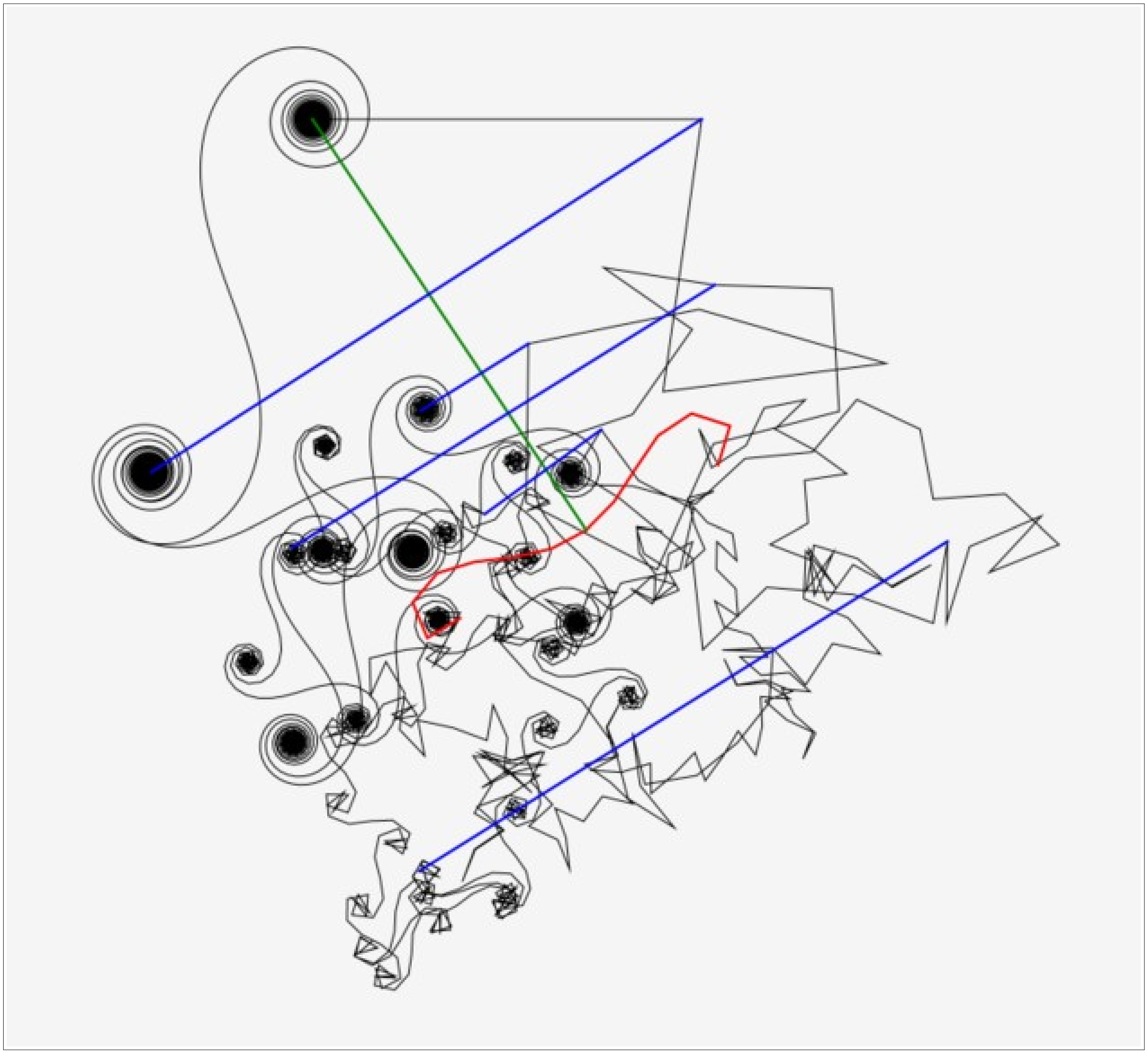
A non-zero value of b. a = 0.5, b = 60000.35, N = 19098. Note the last spiral is not on the origin, and although still parallel the correspondence lines are not perpendicularly bisected by the original symmetry line.

Image 3a



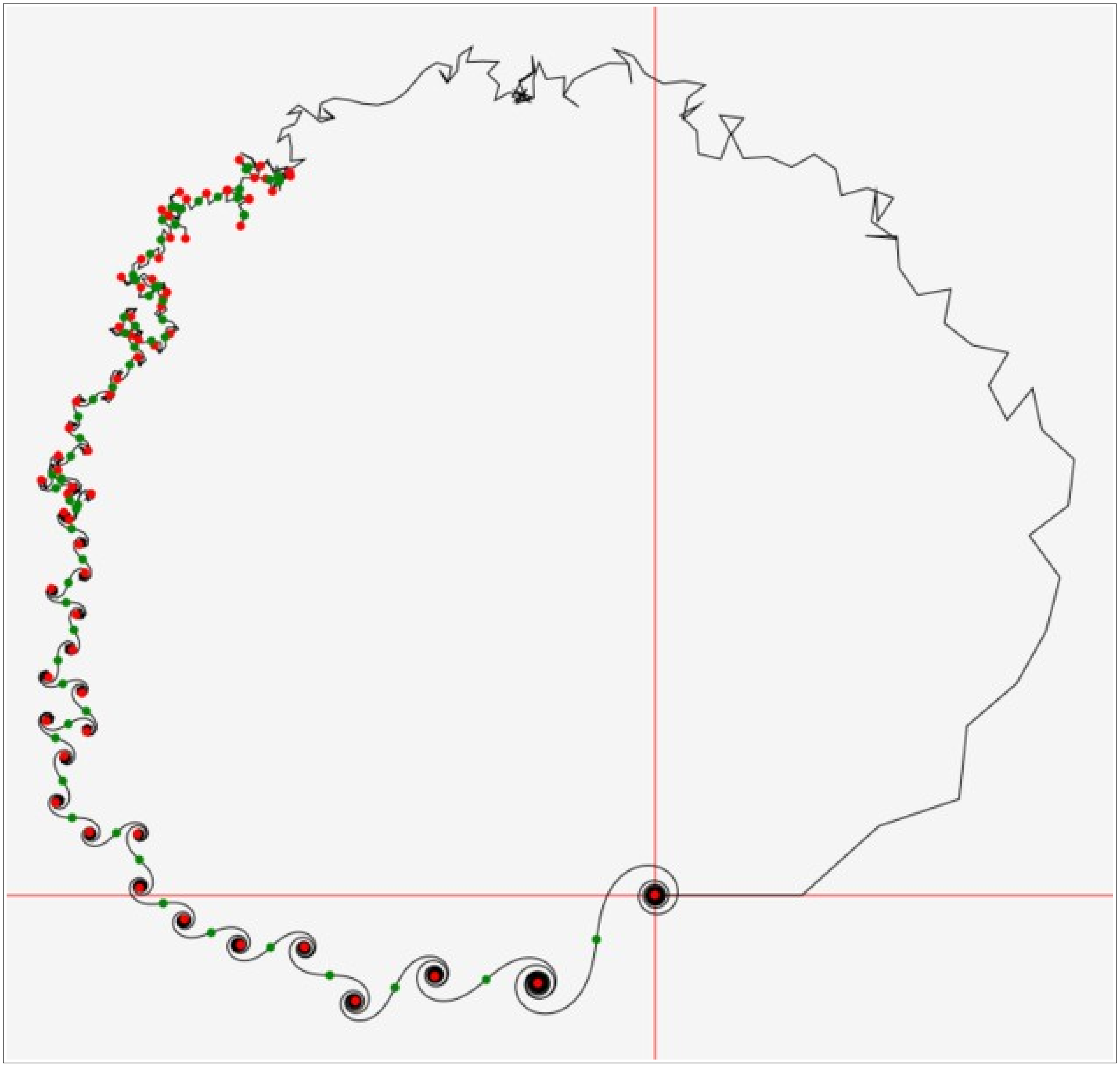
New symmetry line defined by ‘perpendicular bisector of correspondence lines’.

Image 4



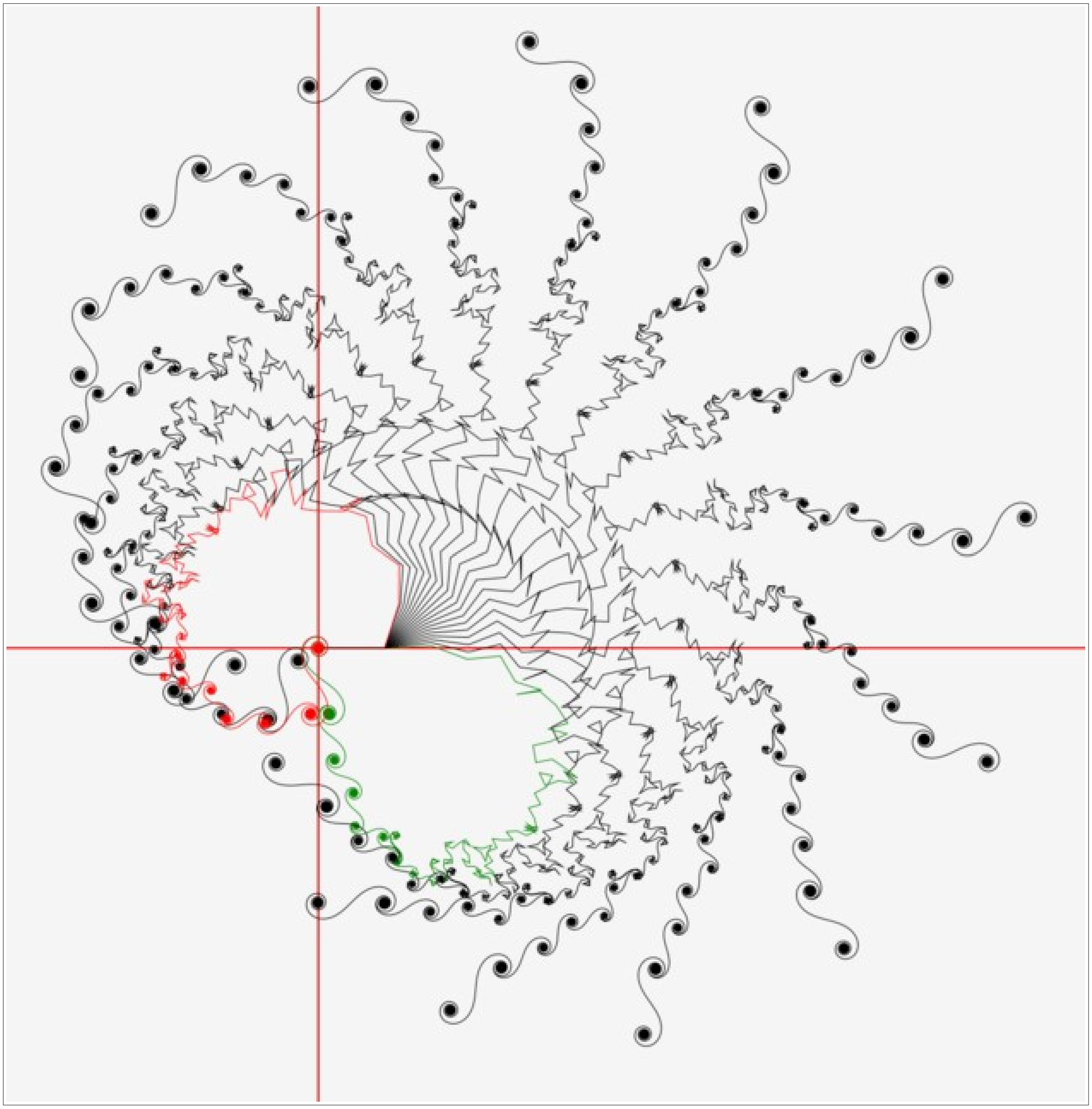
A zero at a=0.5, b=60005.979477112, N=19100, showing that while considerably more tangled than the '60k' example all zeros show the same symmetry and features.

Image 5



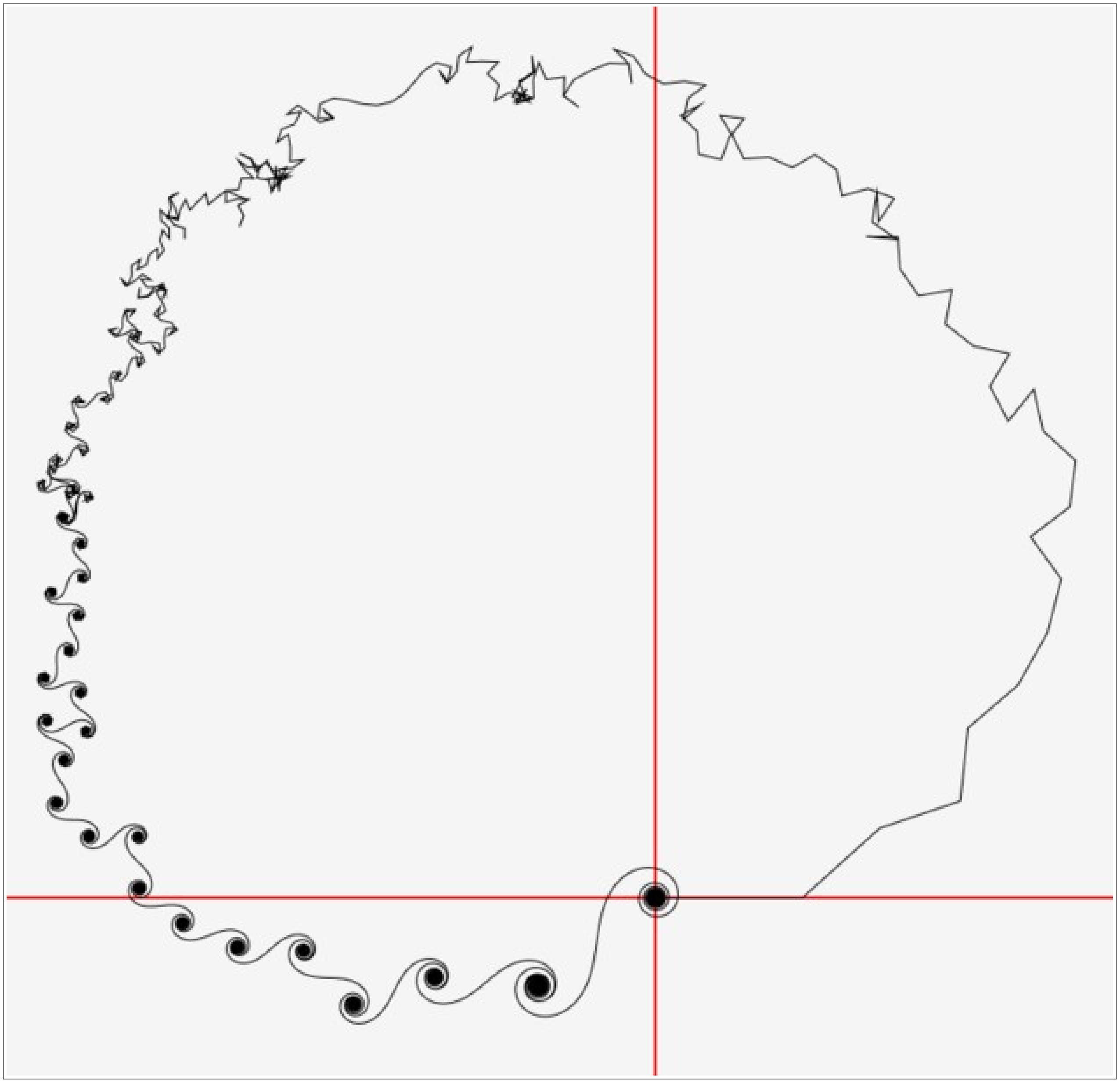
The ‘60k’ showing the inflection points from about n=√(b/π) to n=N=19098. The spiral inflections are red and the medial inflections green.

Image 6



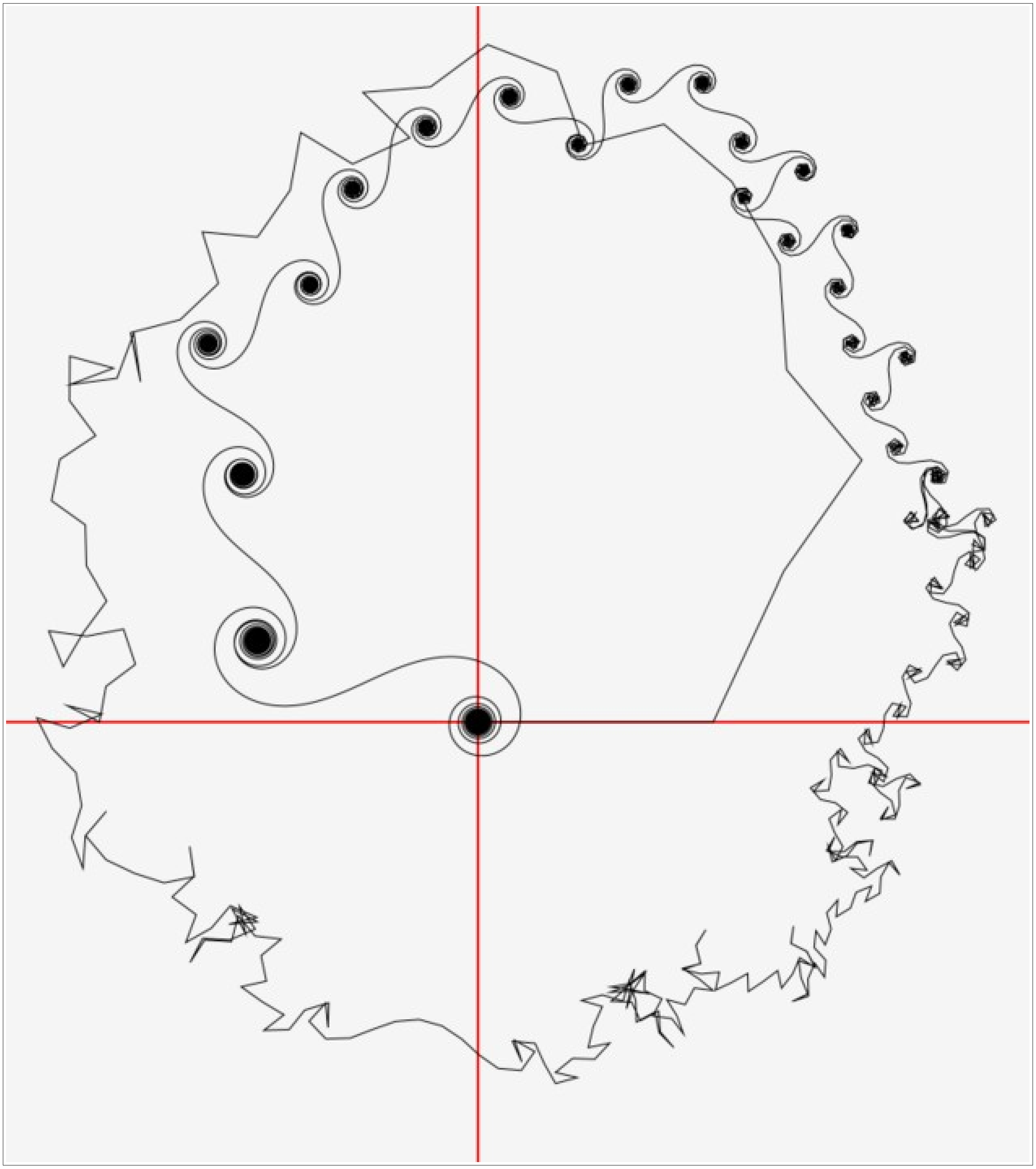
Starting with a=0.5 and b=7034.292019875 in green, incrementing b by 0.1 17 times then one more time with a smaller increment to land on the origin in red at b=7036.042983450. N is recalculated for each plot *to rounded b/π.*

Image 7



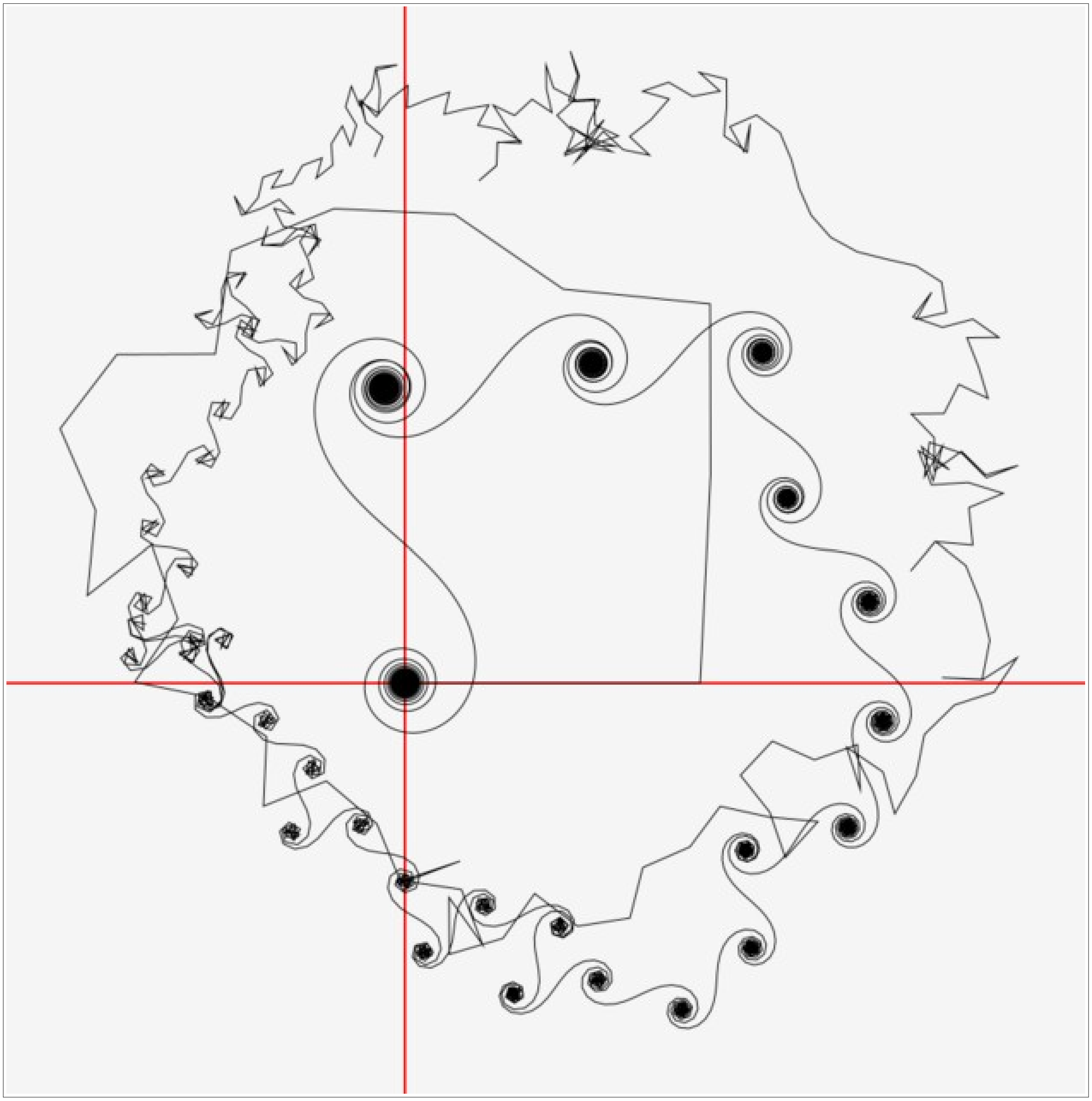
The starting plot *(the '60k') for a series of consecutive zero values. All have a=0.5 and N set to b/π.*

Image 8



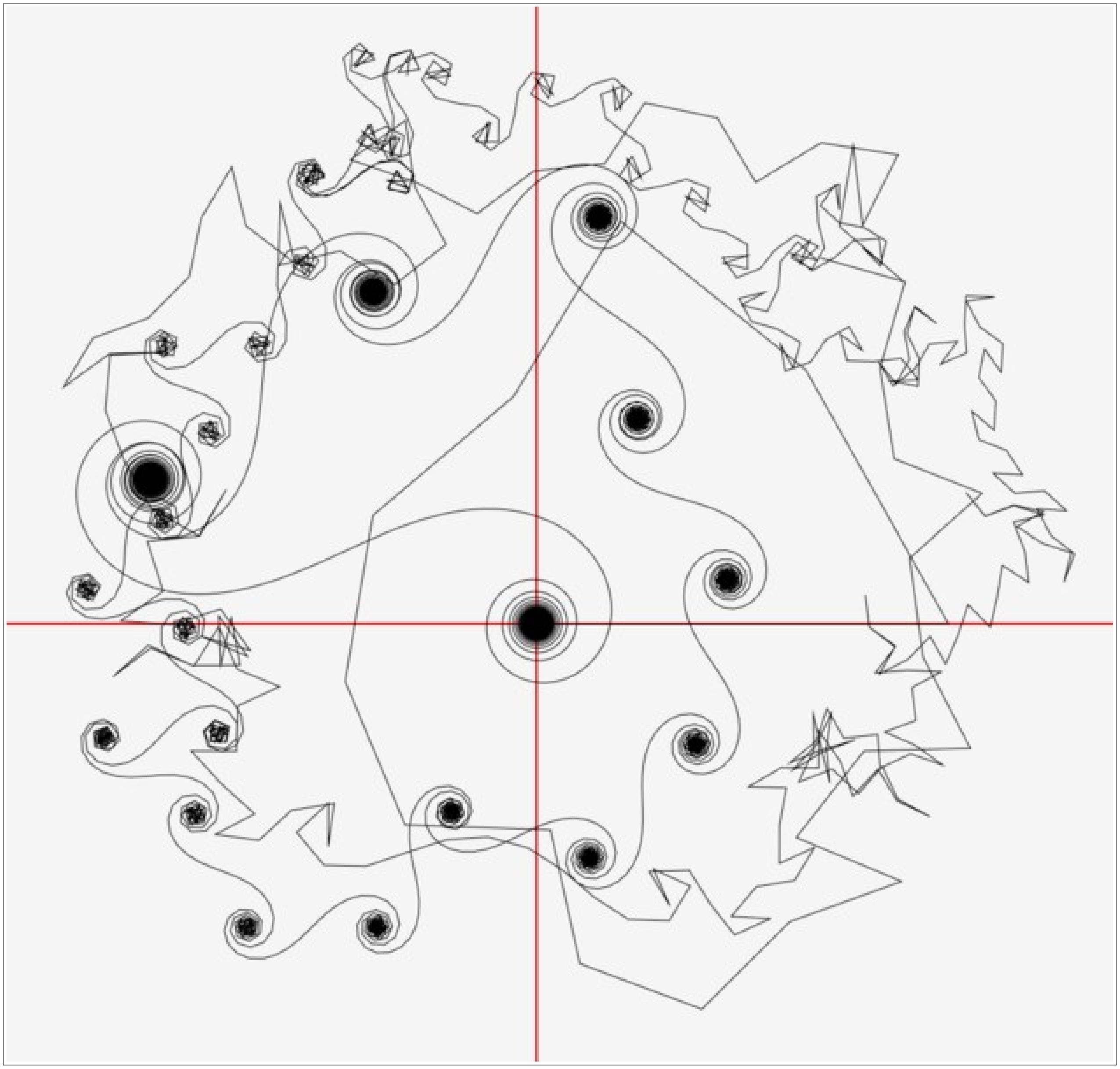
A zero at b=60001.017592093. Note that there are now two distinct loops.

Image 9



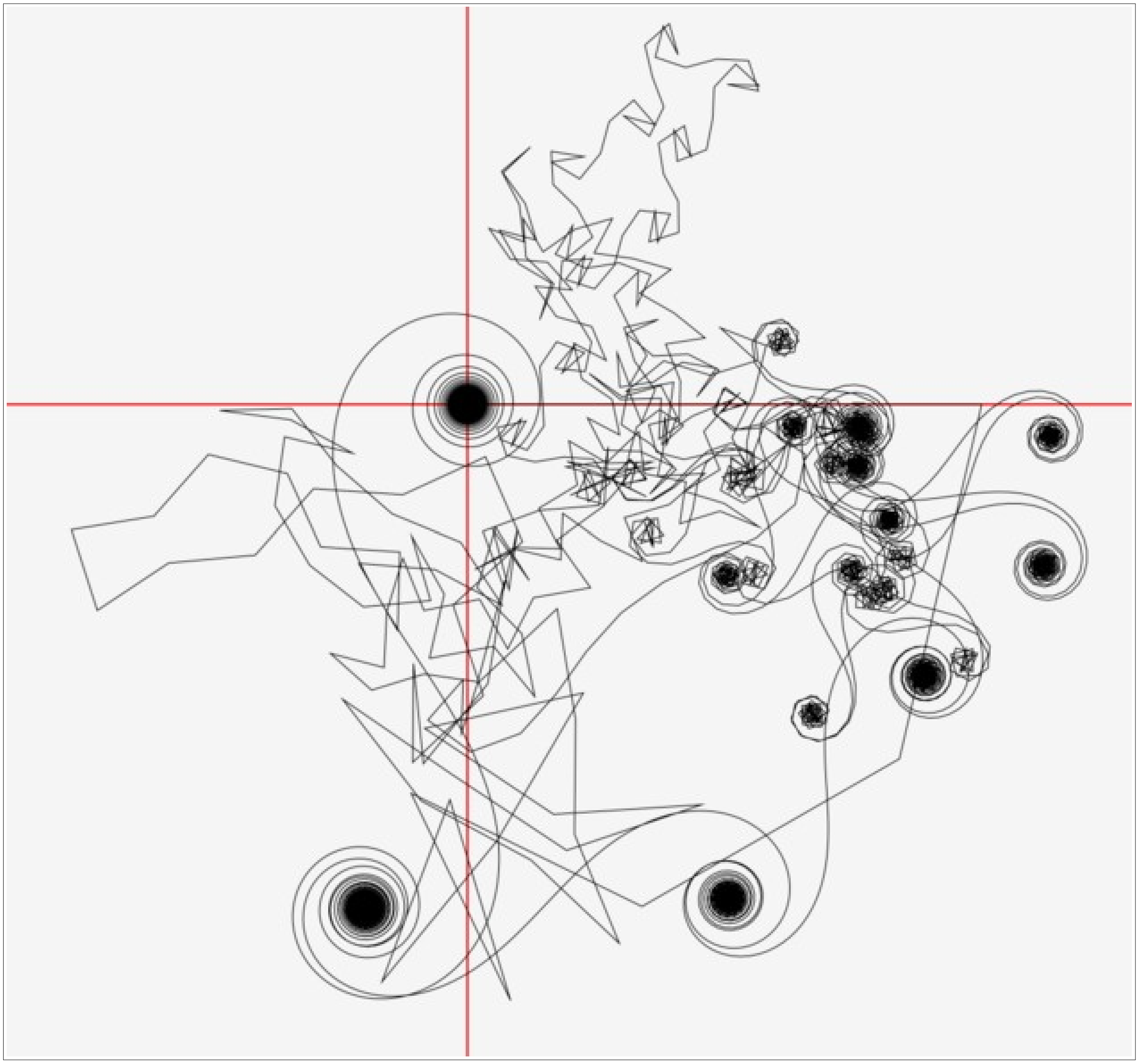
The zero at b=60001.577675896; there are now three loops.

Image 10



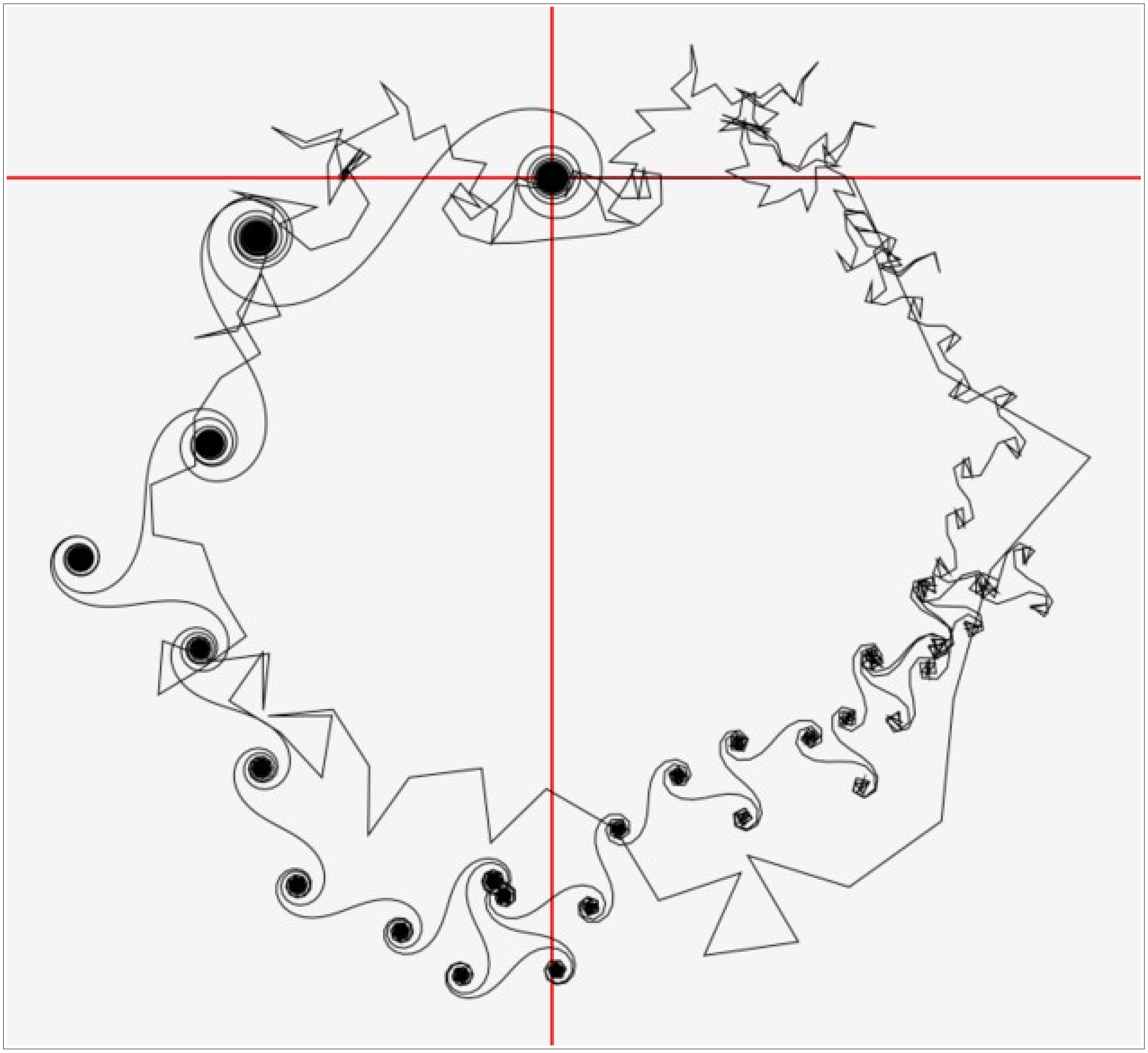
The zero at b=60002.388346887, with four loops.

Image 11



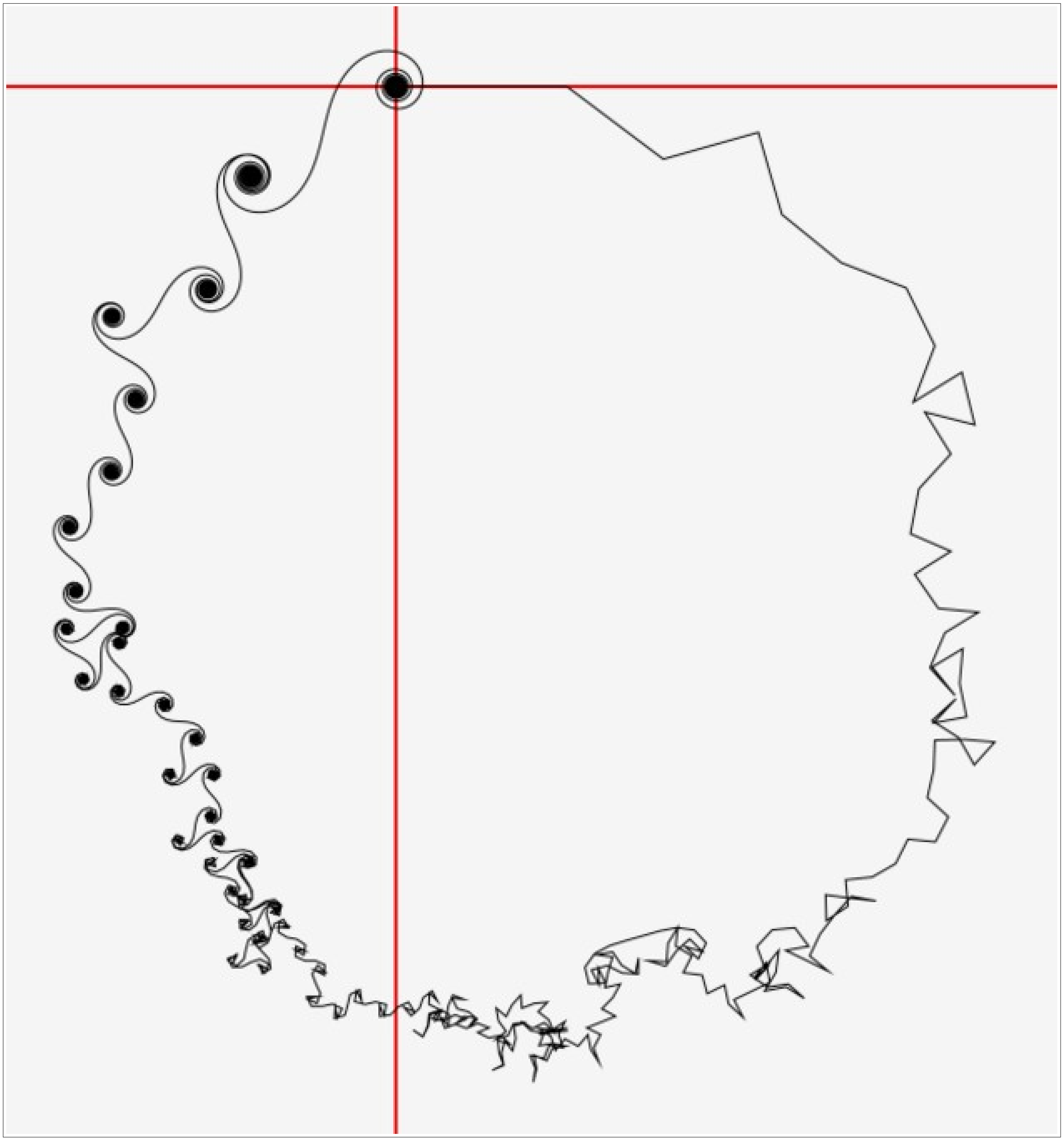
Skipping to the 22nd zero from the '60k', b=60014.917789703. Twenty-two loops, maybe?

Image 12



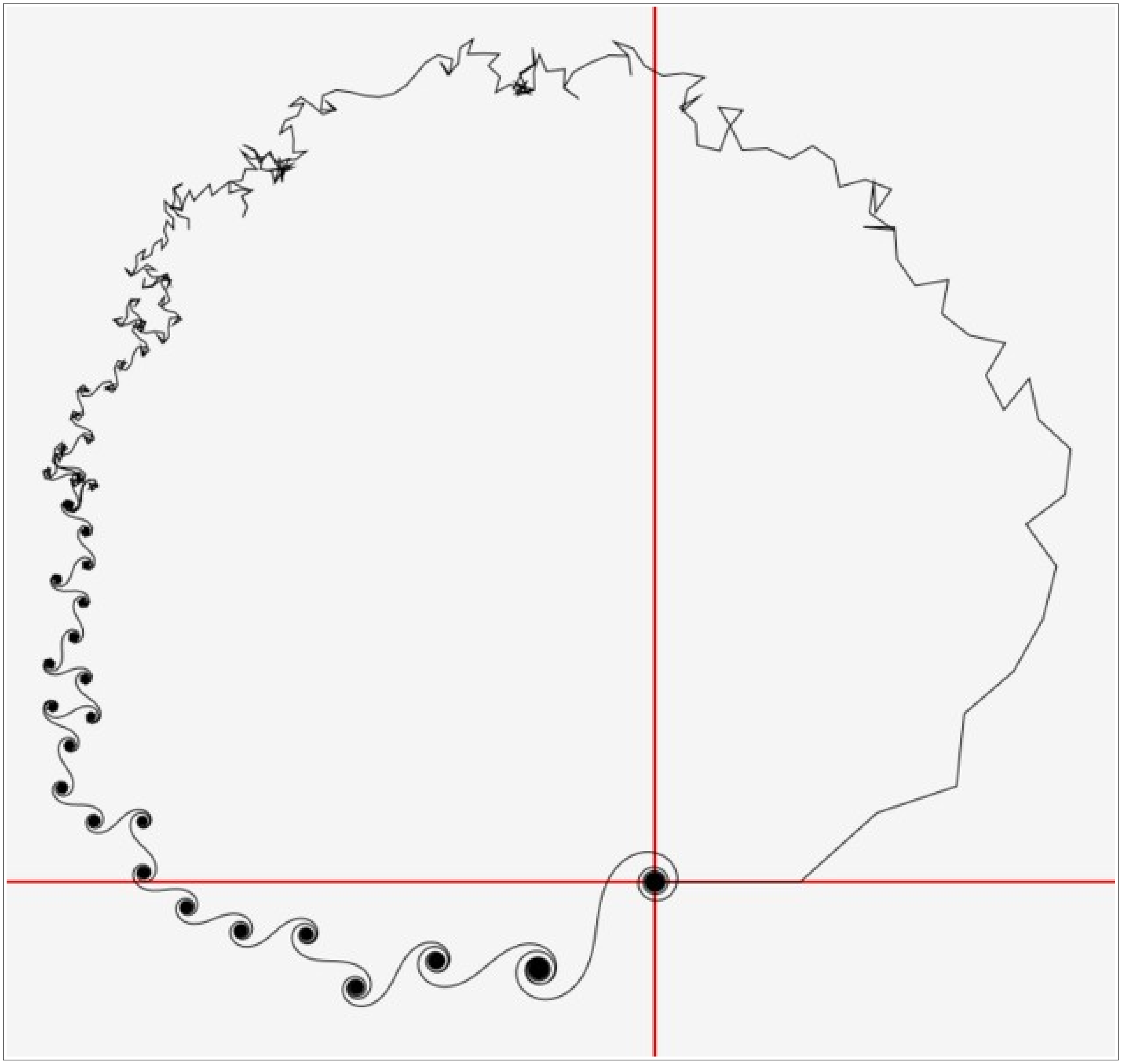
The zero 50 zero values larger than the starting '60k'. It has unwound from a tangled mess back to just two loops. The b value is b=60033.986550311.

Image 13



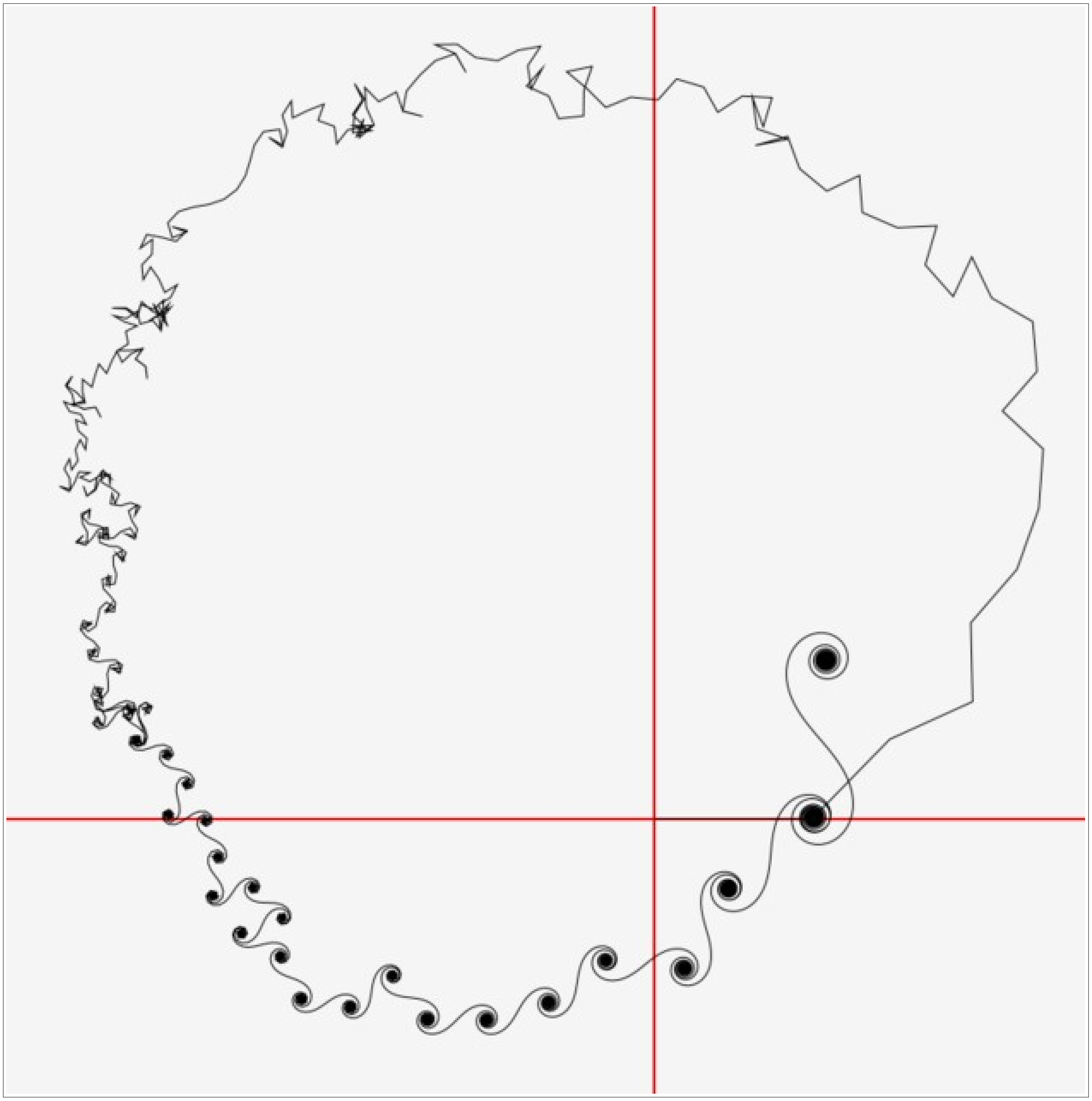
Some 51 zeros after the '60k', at b=60034.711149844, the plot *has completely unwound back to a single loop. This winding unwinding behavior repeats over and over.*

Image 14



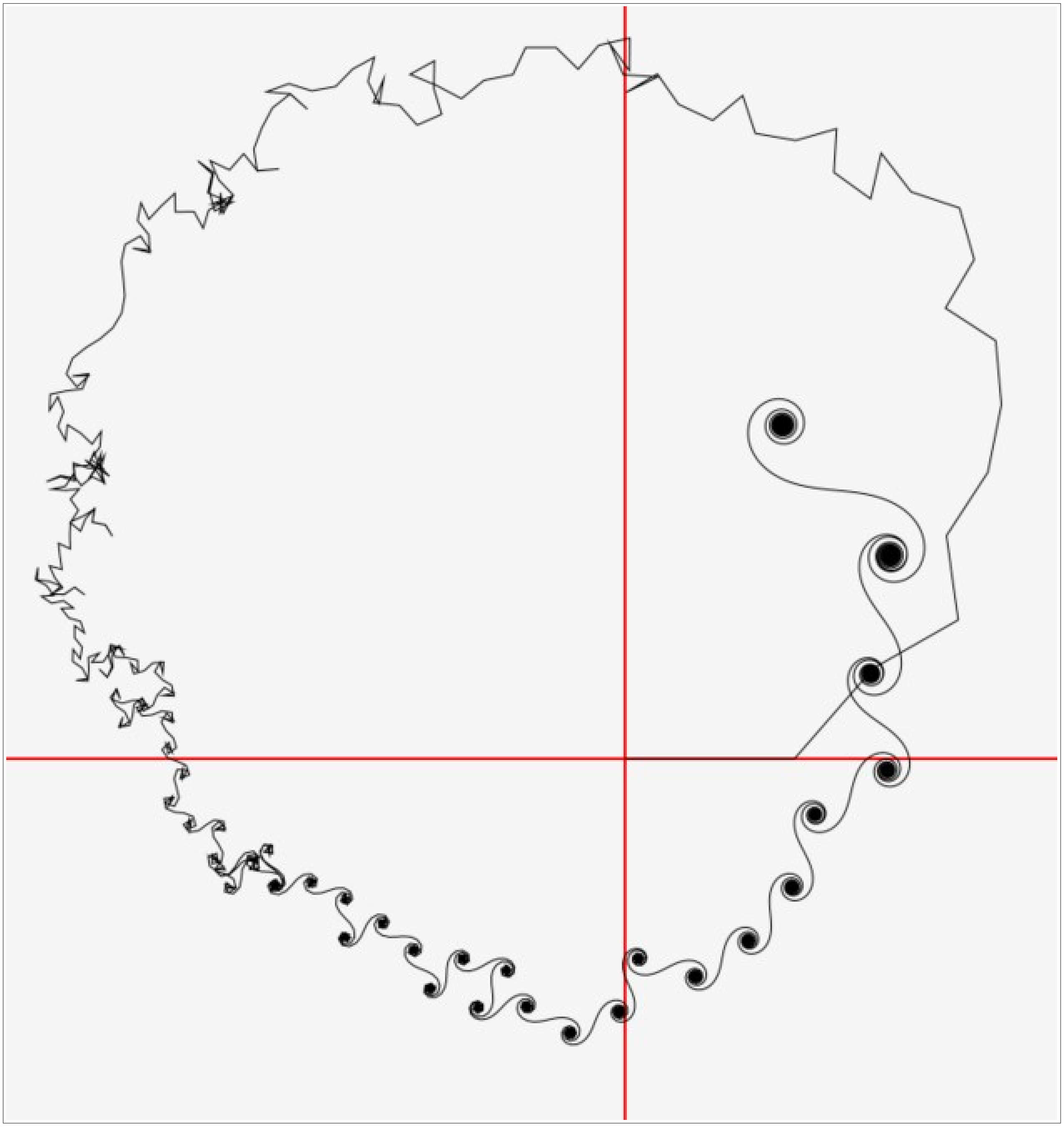
The '60k' with the last spiral inflection point at b/π is coincident with the starting point.

Image 15



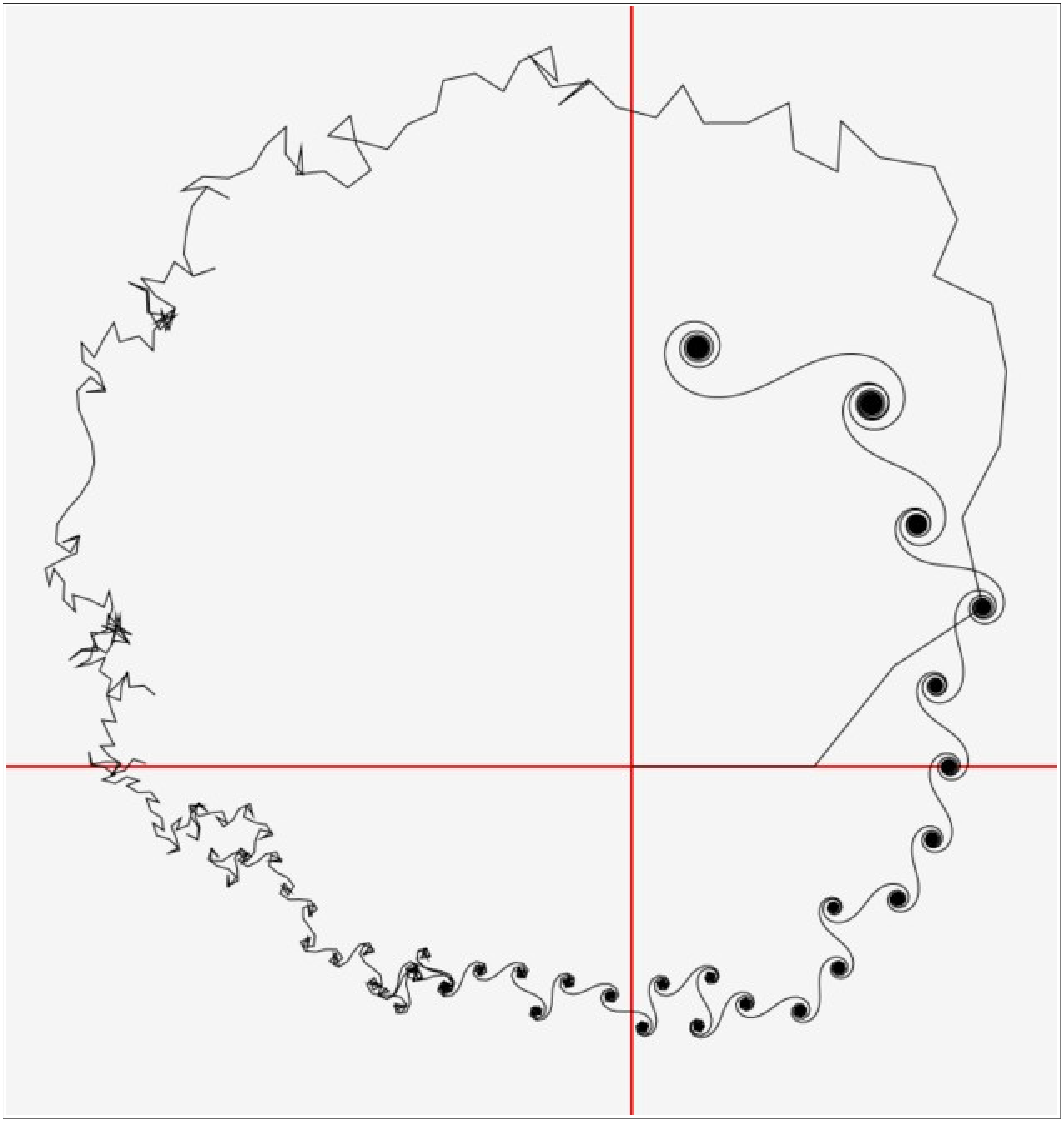
The spiral at b/3π is coincident with point 1. The actual value of b is b=60000.532... . Since the increment is 0.001 the graphical accuracy is just to three decimal places.

Image 16



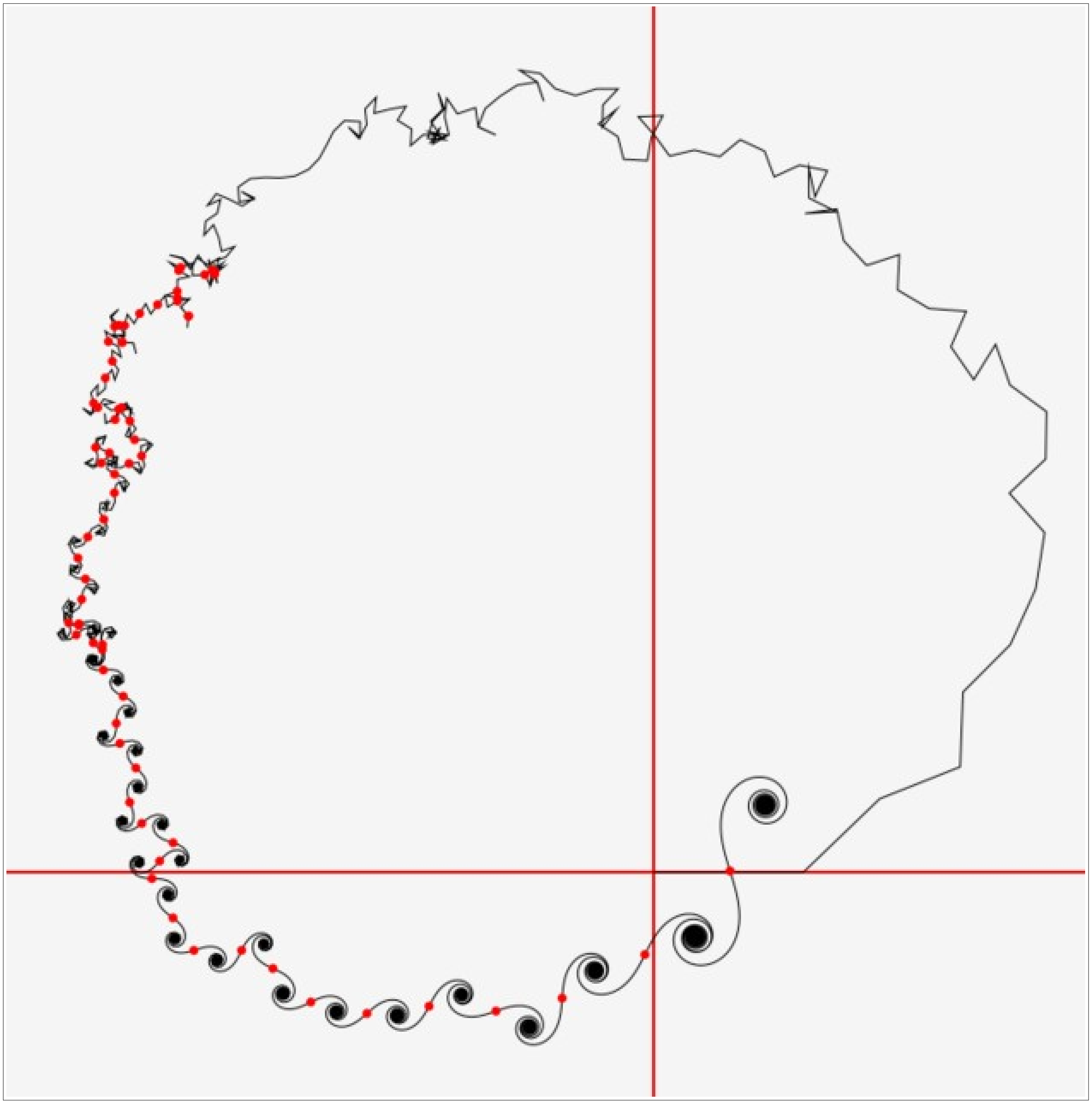
At b=60000.619... the third spiral at b/5π is coincident with the 2nd point.

Image 17



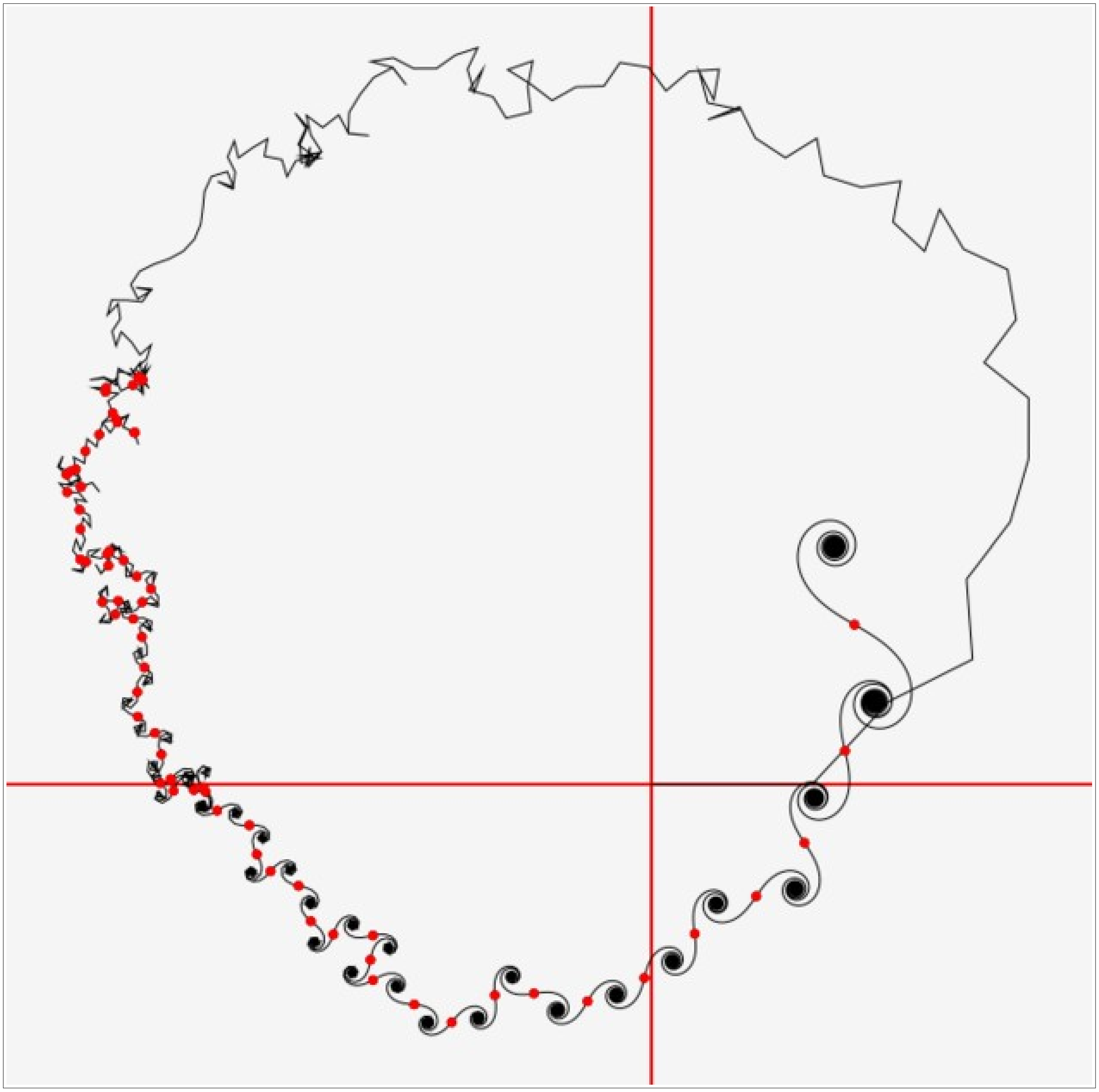
This correspondence / coincidence continues until it becomes too complicated to descern. Here the 4th spiral at b/7π is coincident with point 3, at b=60000.679... .

Image 18



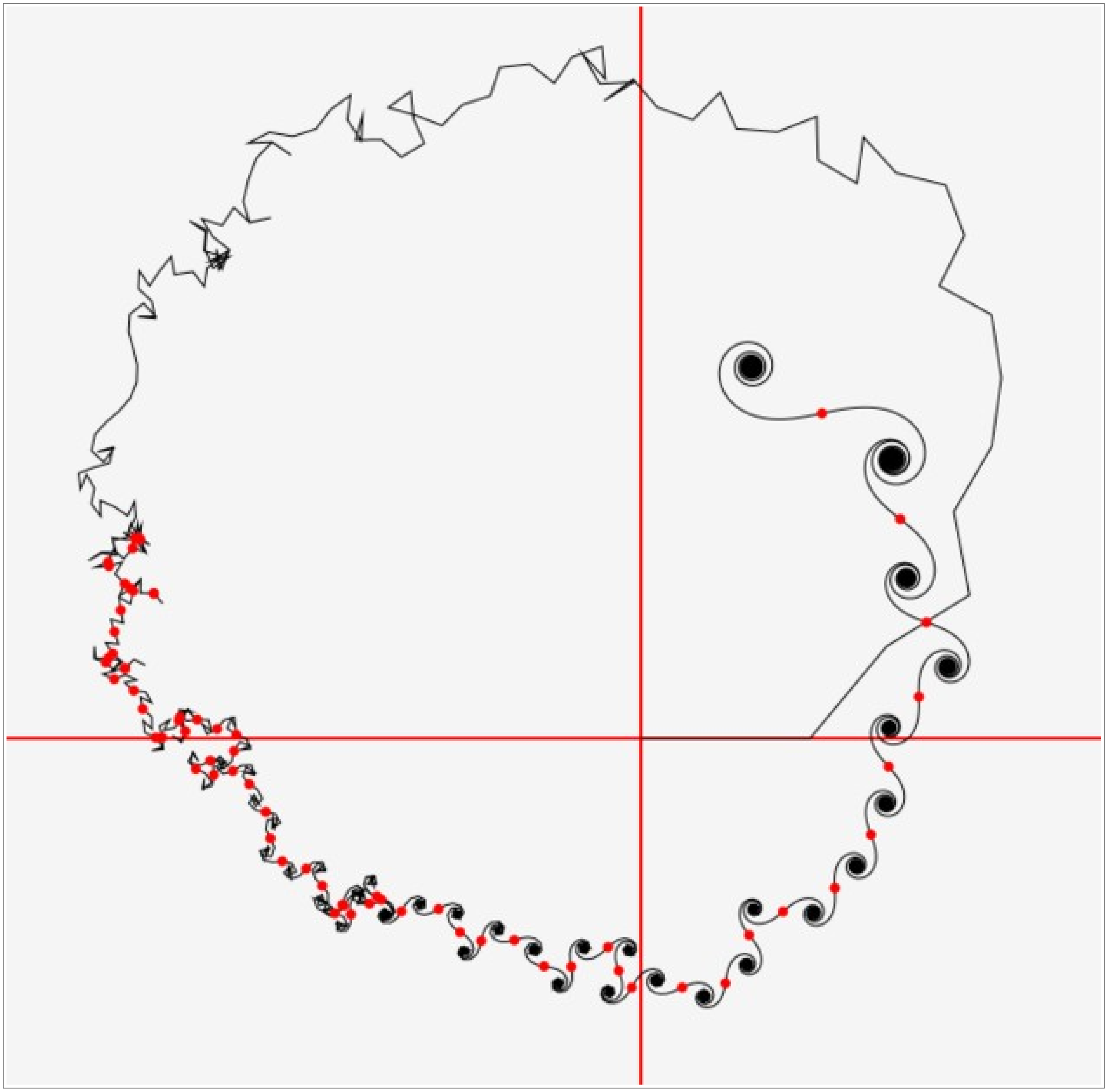
The medial inflection points are shown in red. The last medial inflection point at b/2π is coincident with the midpoint of the first vector.

Image 19



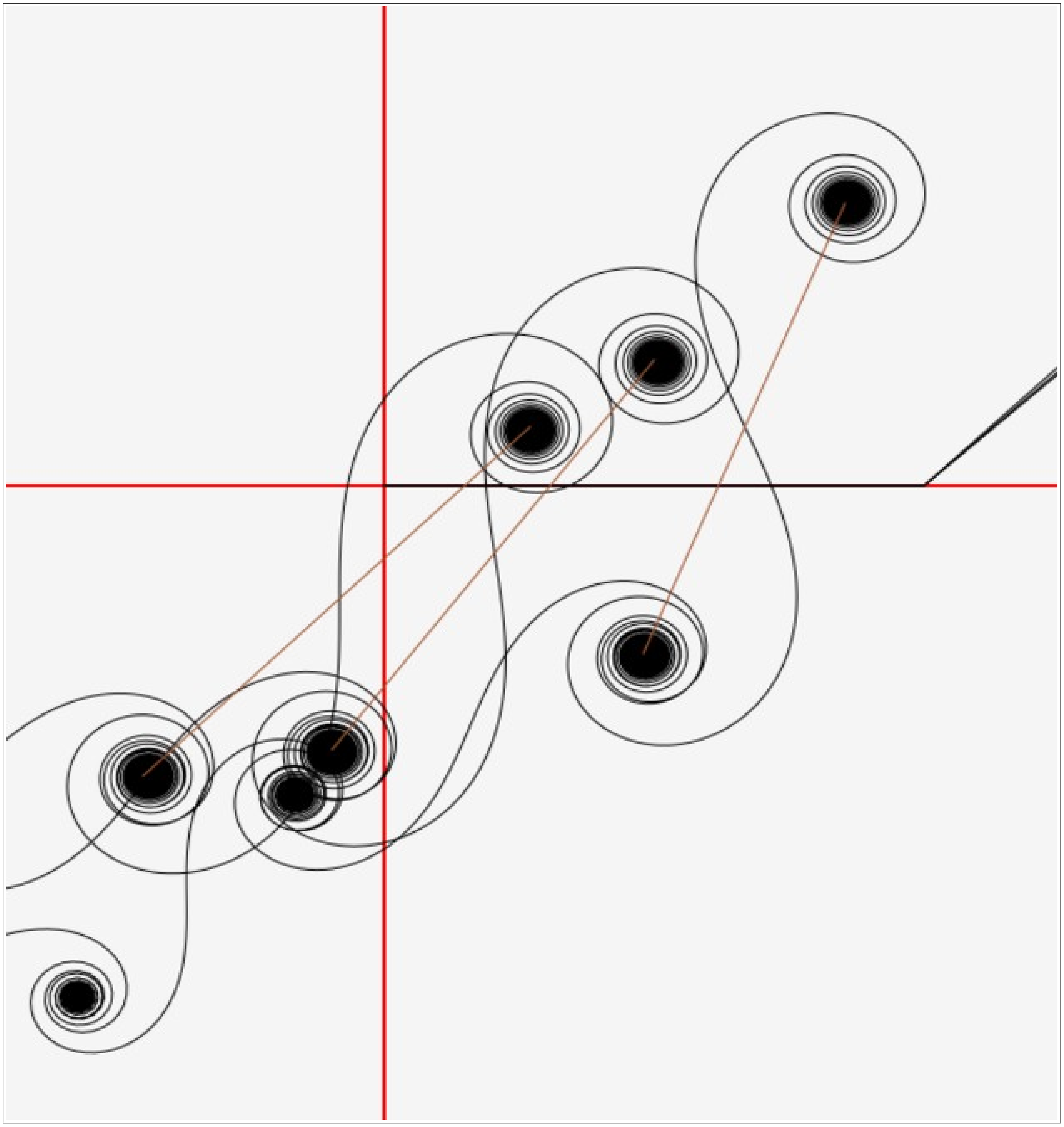
Here the second to last medial inflection point is coincident with the midpoint of vector 2.

Image 20



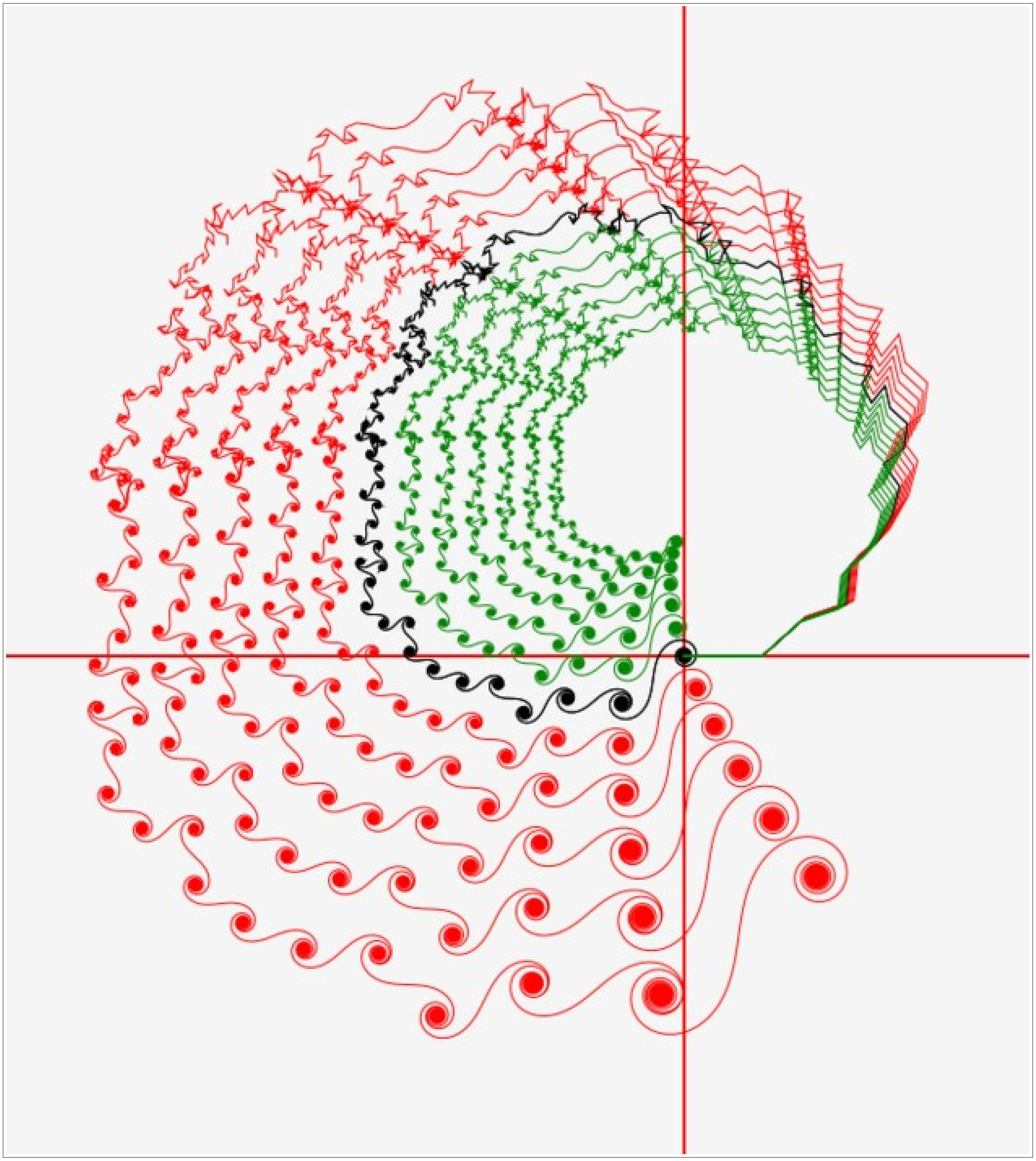
And here the third to last medial inflection point is coincident with the 3rd vector's midpoint. These coincidences continue as the value of b is increased.

Image 21



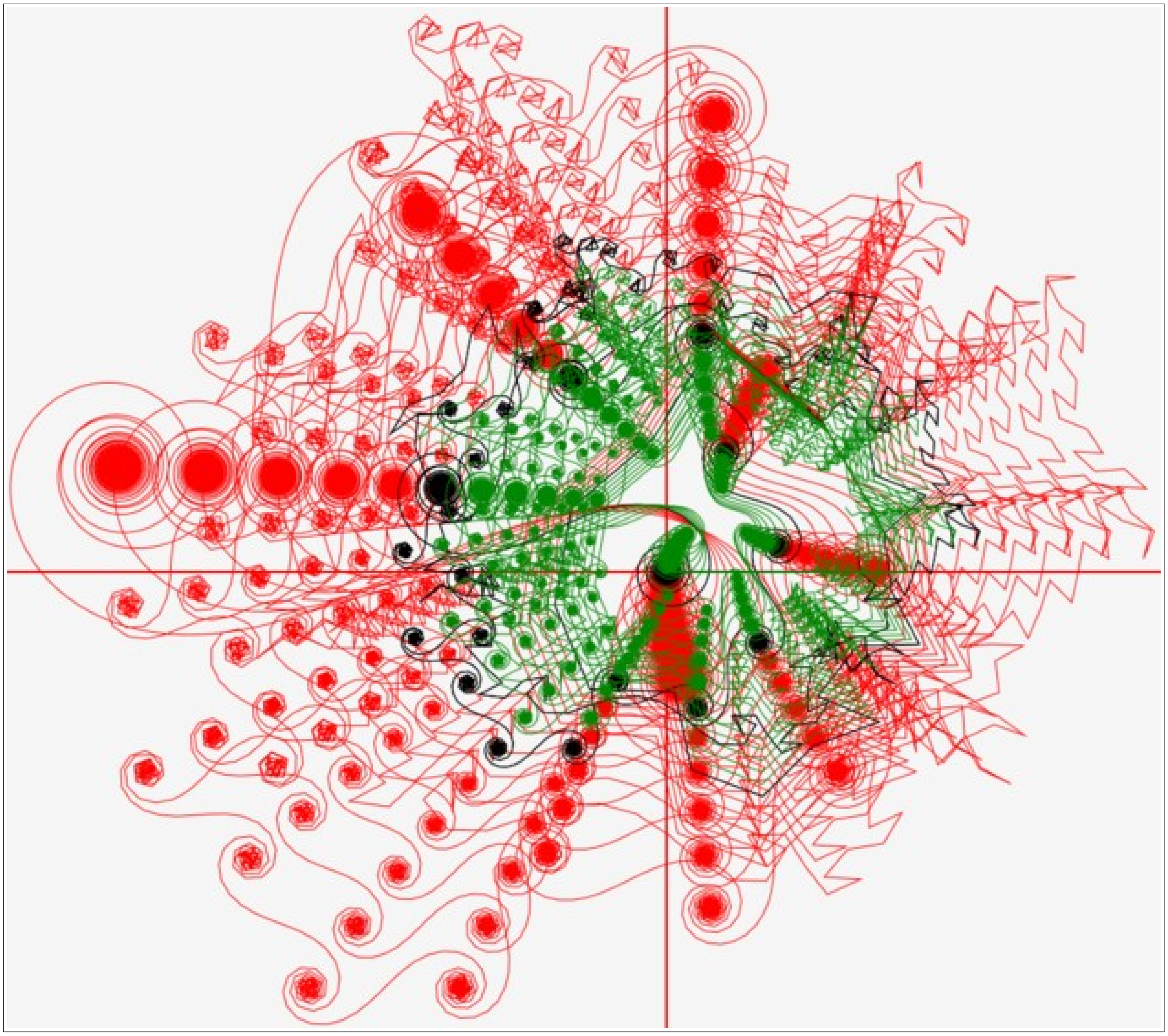
Closeup of the coincident aspects of vector 1 with a line drawn between the last two spirals (brown). This effect is displayed for all vectors and lines between spirals. Note that the midpoint of the brown line is at the midpoint of the spiral line between the two spirals, which is the medial inflection point.

Image 22



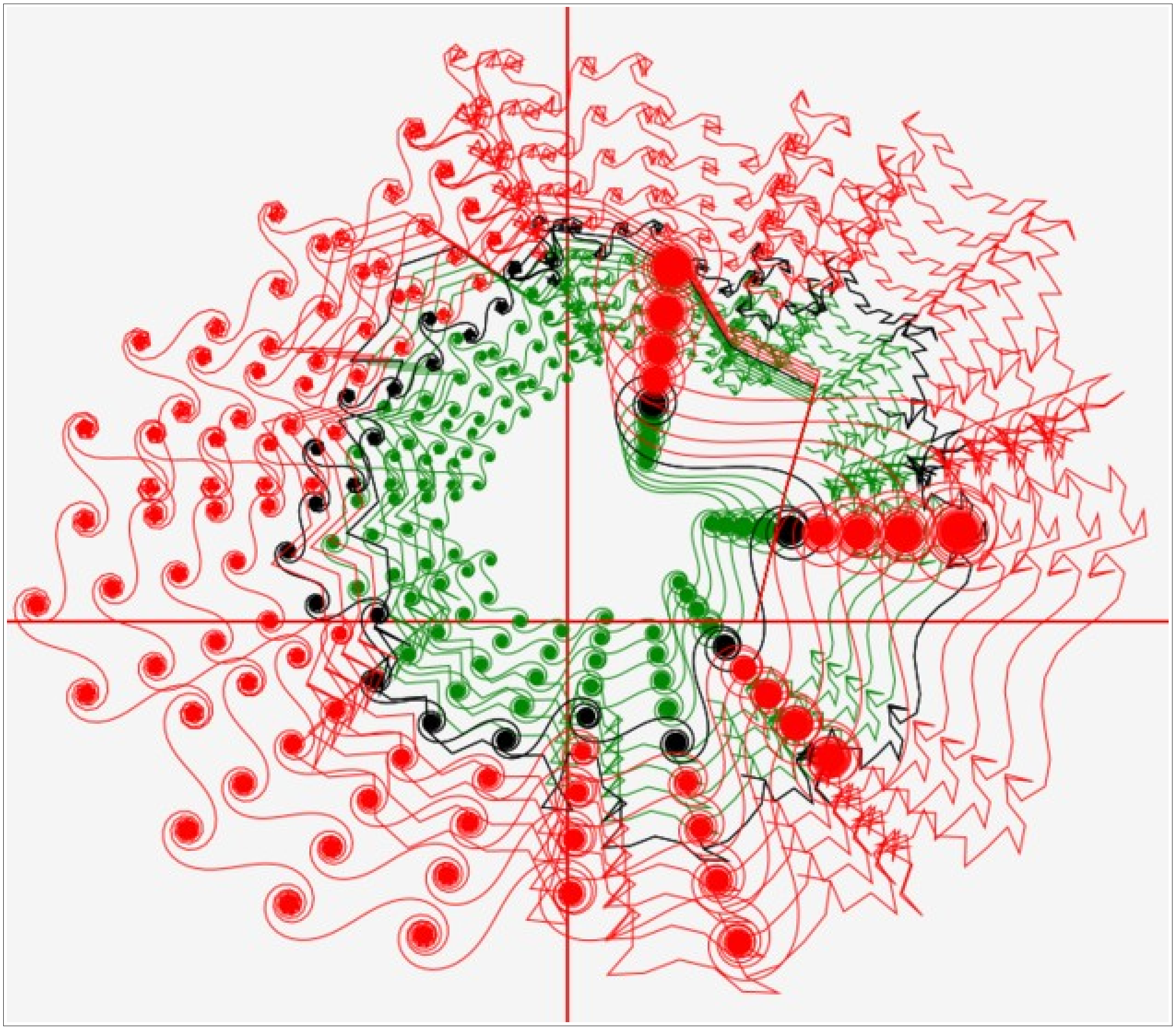
The '60k' with 0.4 < a < 0.62. The zero value, 0.5, is shown in black.

Image 23



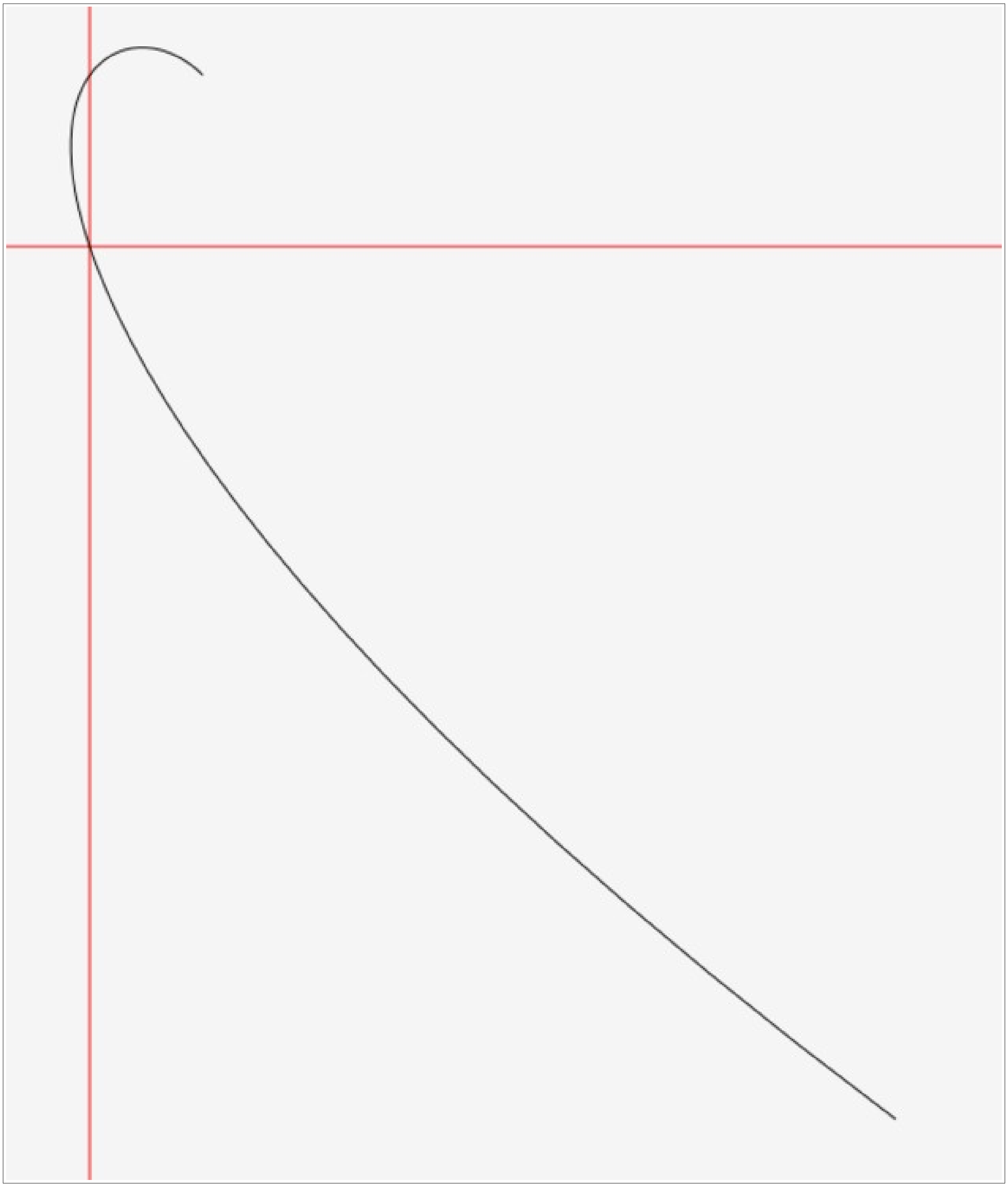
Varying 'a' from a zero of b= 60002.388346887.

Image 24



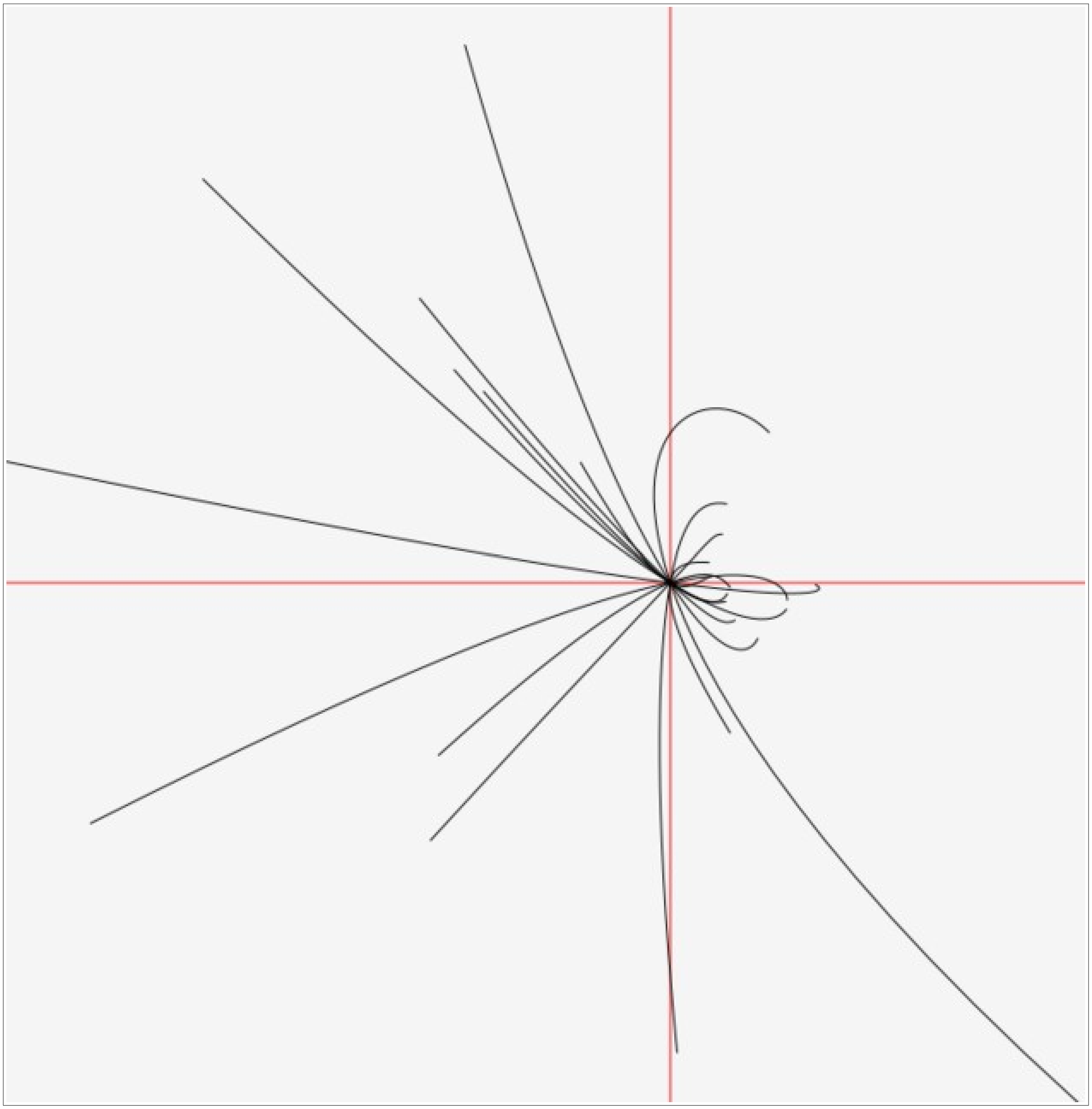
A non-zero with b=60001.31759209299 shows the same response to altering the ‘a’ value.

Image 25



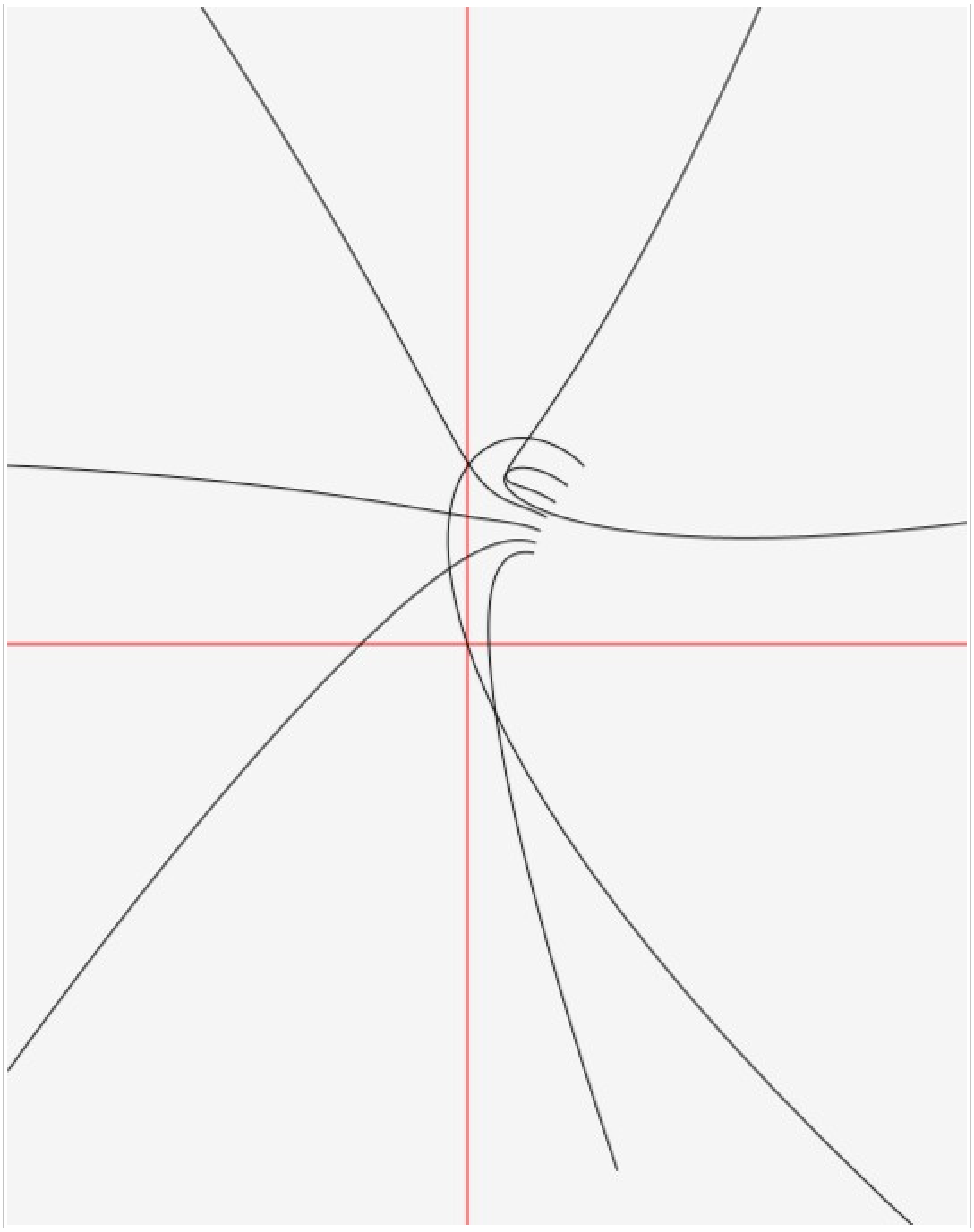
The ‘60k’: b=60000.441207279, 0<=a<=1. Track of last inflection point. a=0.5 at origin.

Image 26



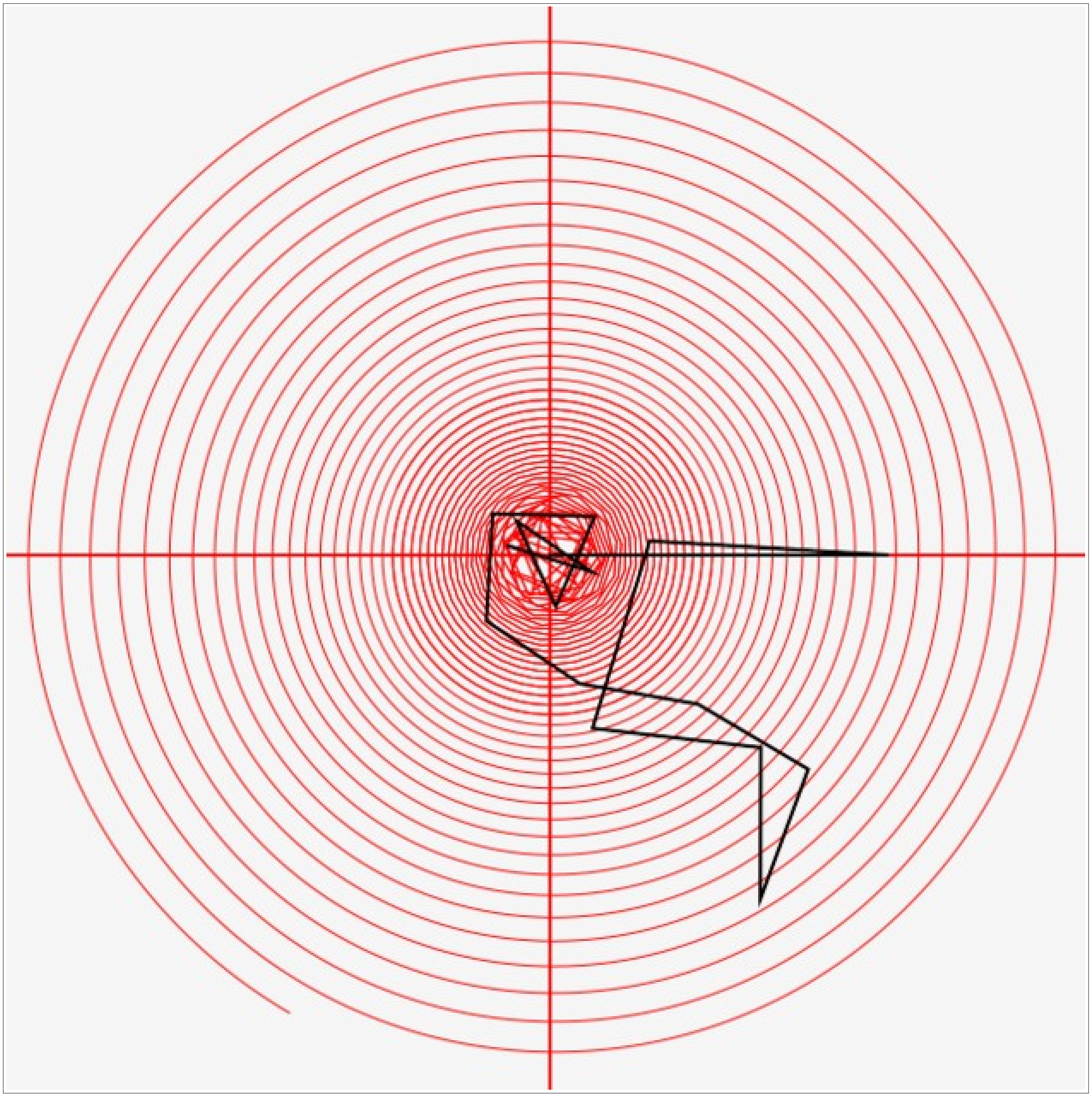
Thirteen zeros starting with ‘60k’, tracks of last inflection 0.3<=a<=1.

Image 27



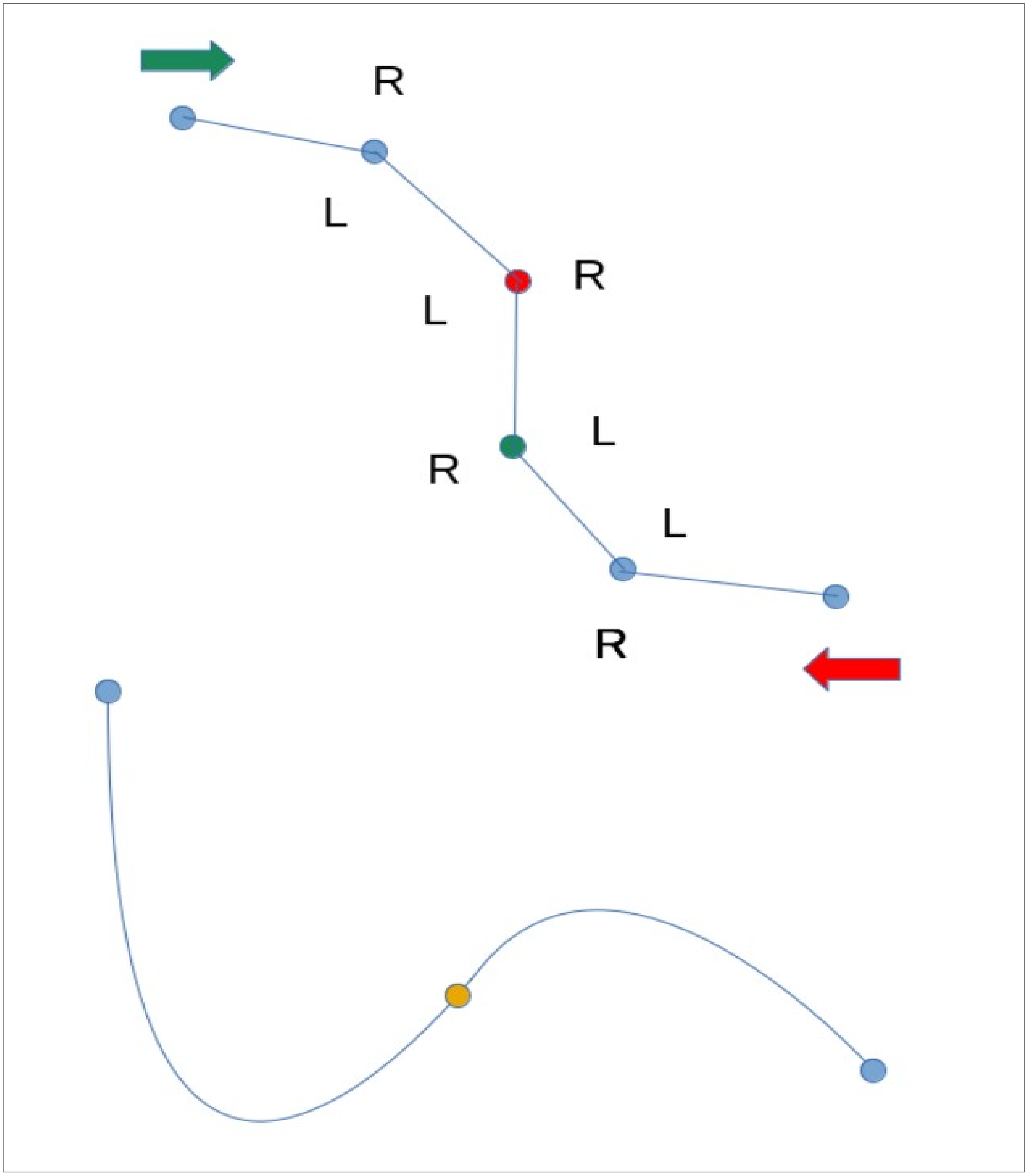
Non-zeros: ‘60k’<=b<=60001, 0.3<=a<=1.

Image 28



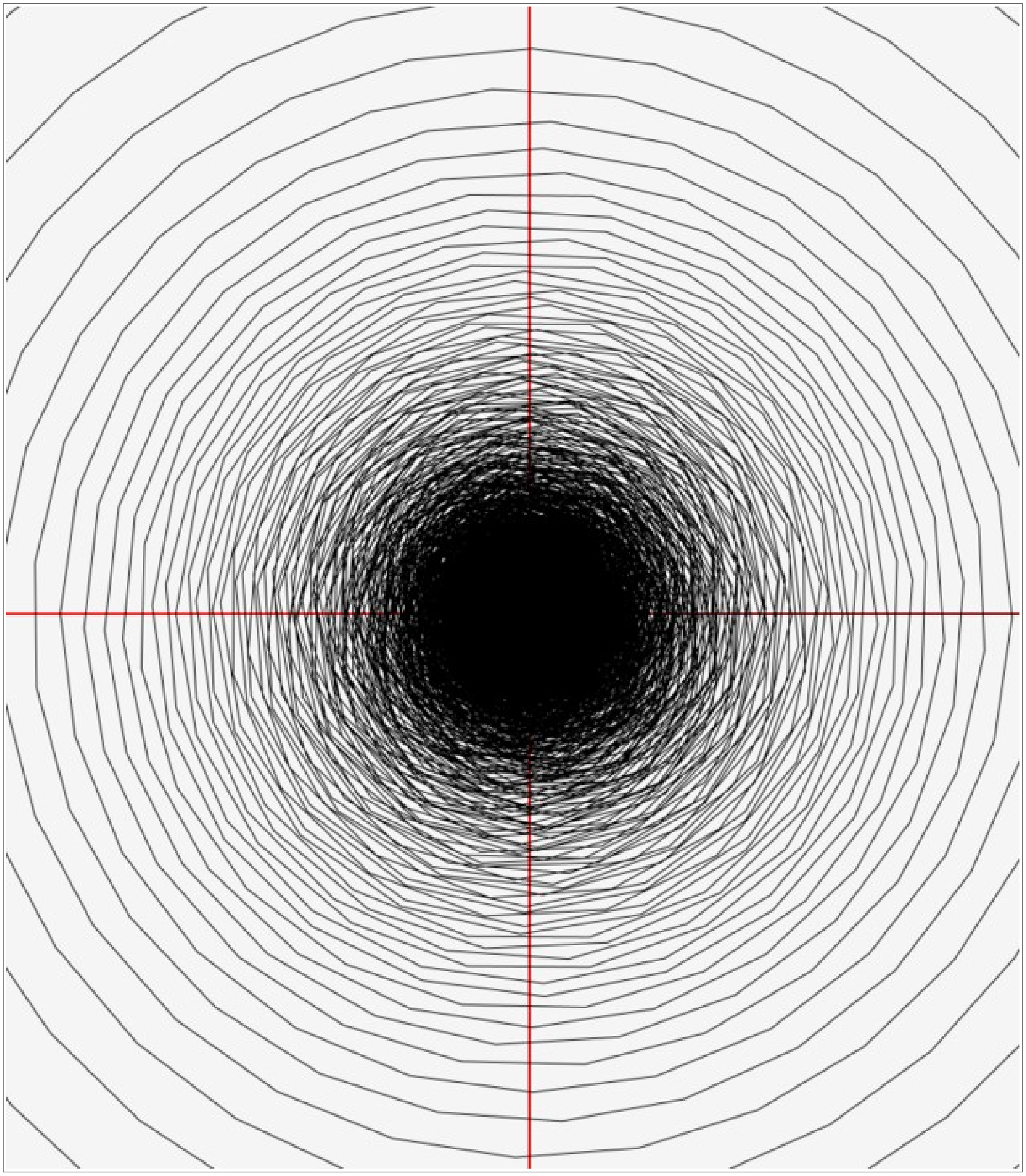
The a=0.5, b=49.773832478, N=15 Riemann zero in black, with ‘n’ values extended to 6000 in red.

Image 29



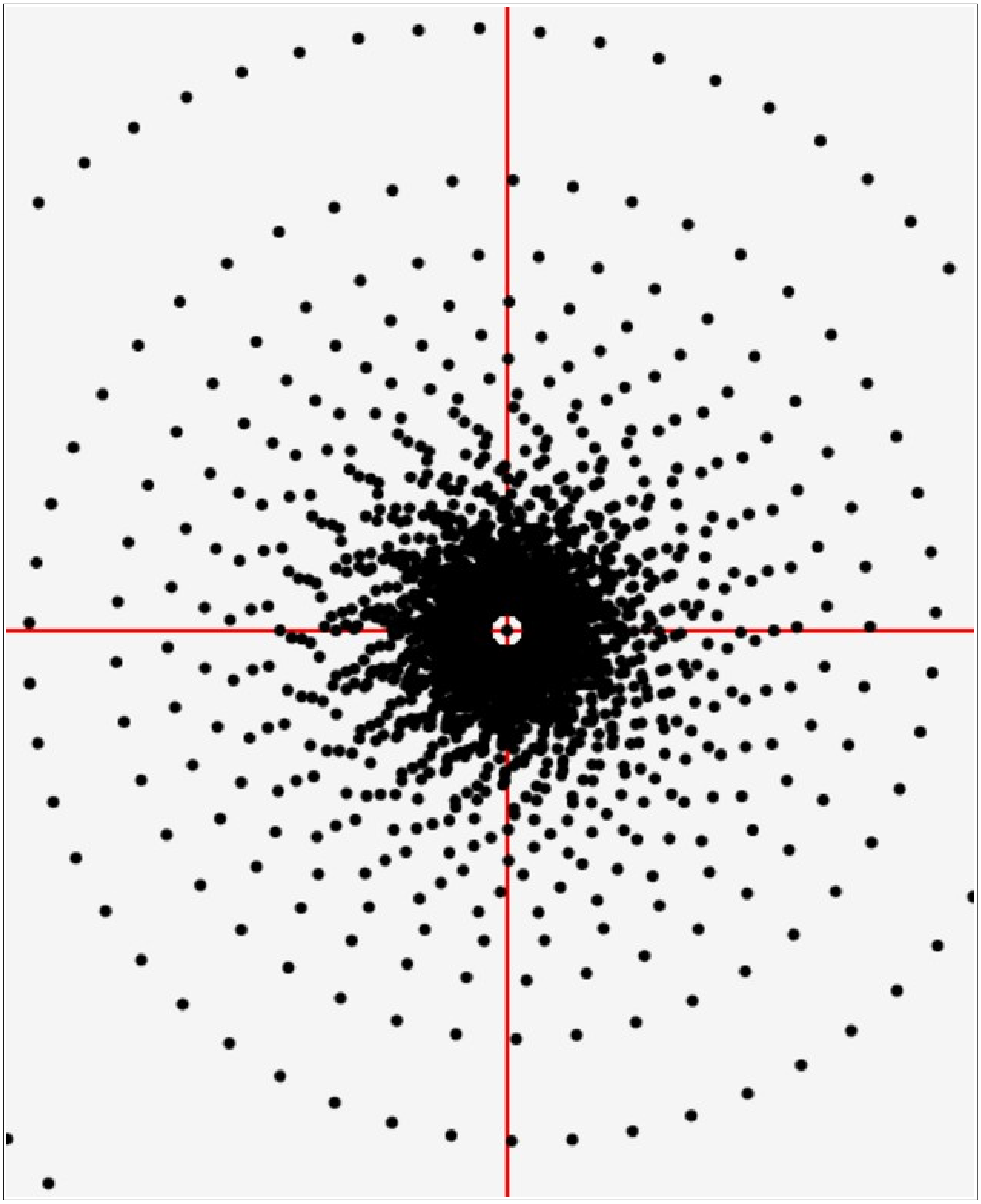
The difference between discrete (top) and continuous (bottom) inflection points.

Image 30



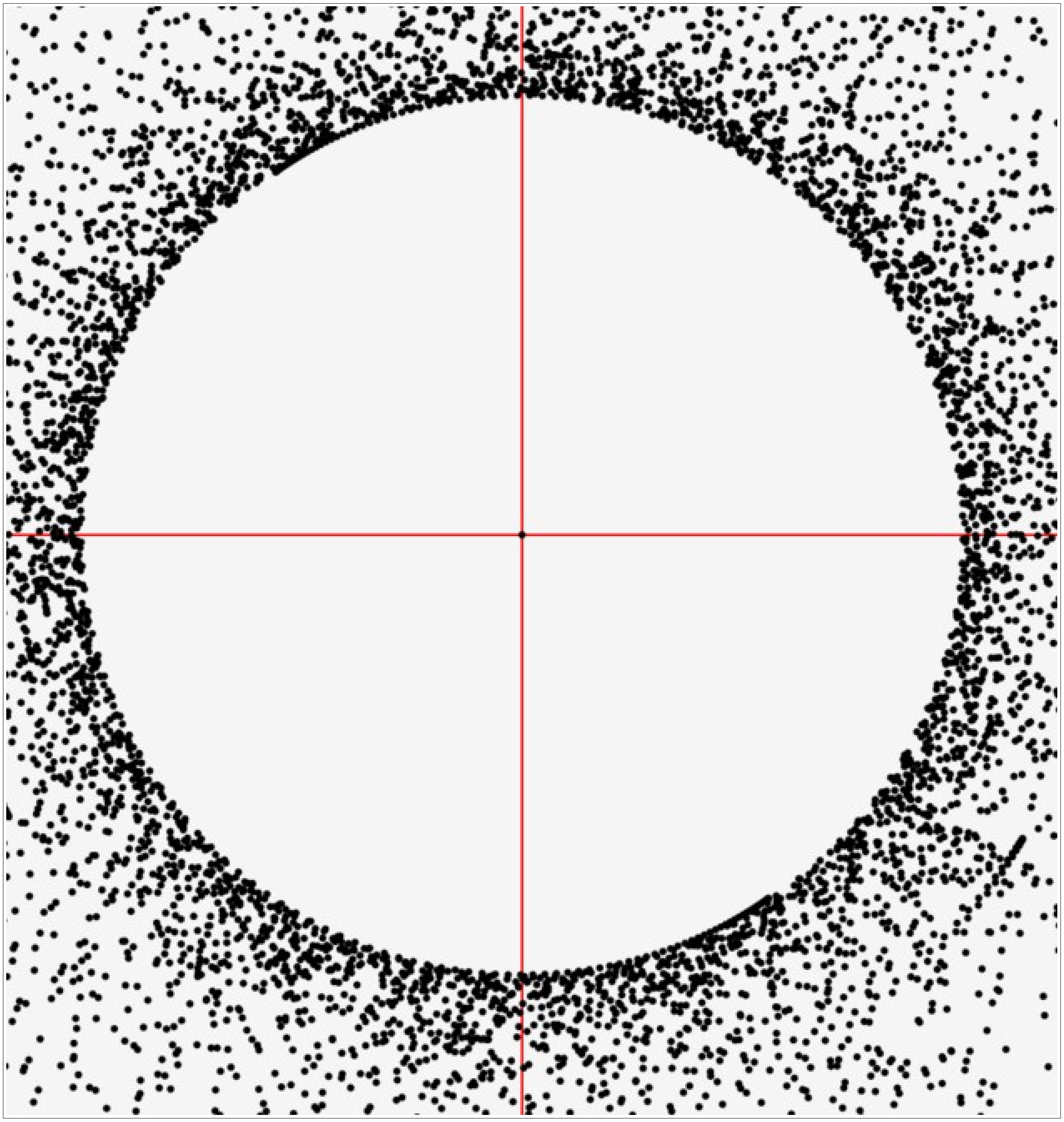
'60k' last spiral at magnification 10,000.

Image 31



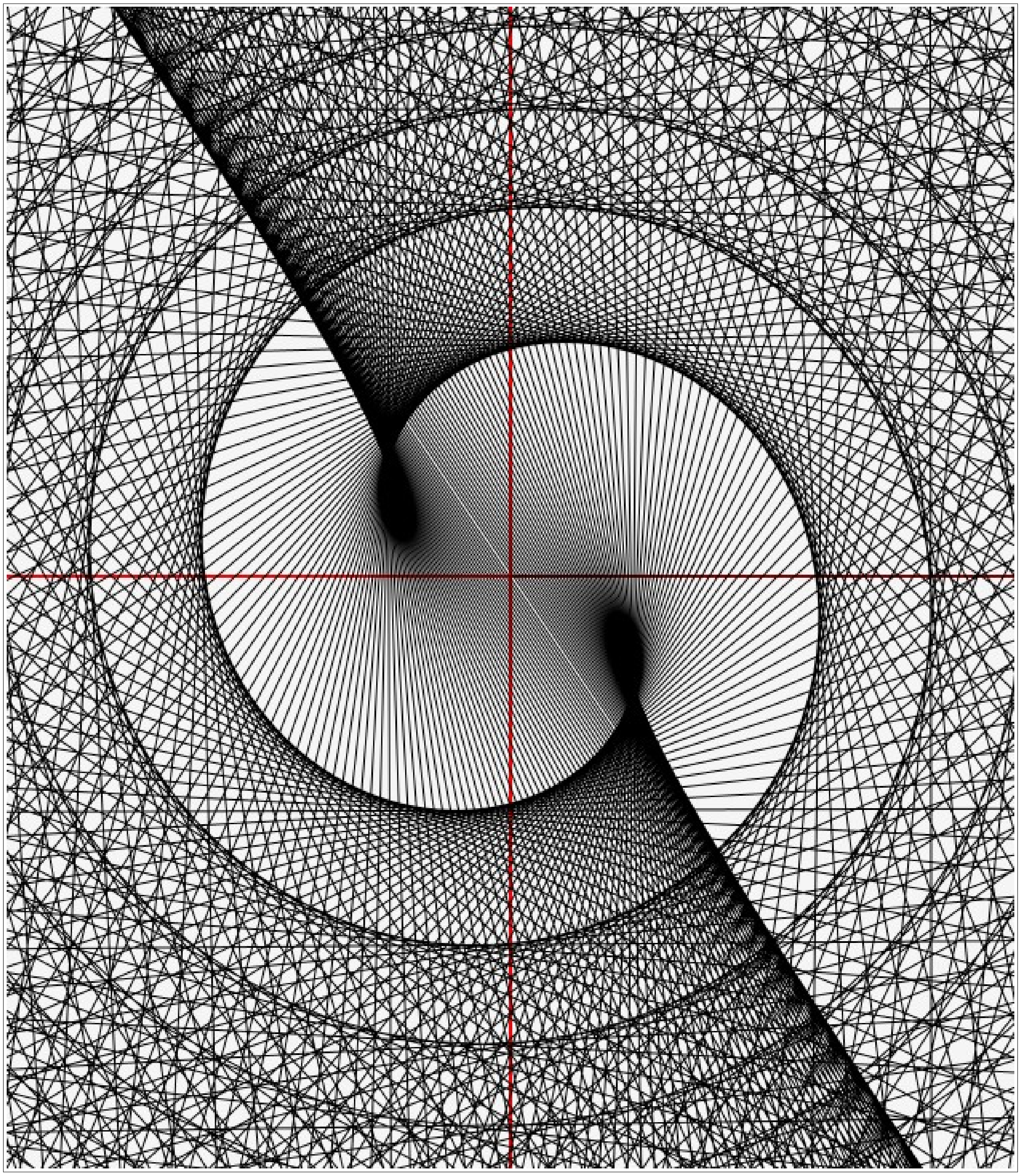
'60k' at a magnification of 3000 displaying the victor addresses as dots, not vectors. Note the hole in the center. The single dot in the center is the starting dot (term n=0) and is only the starting point on the origin.

Image 32



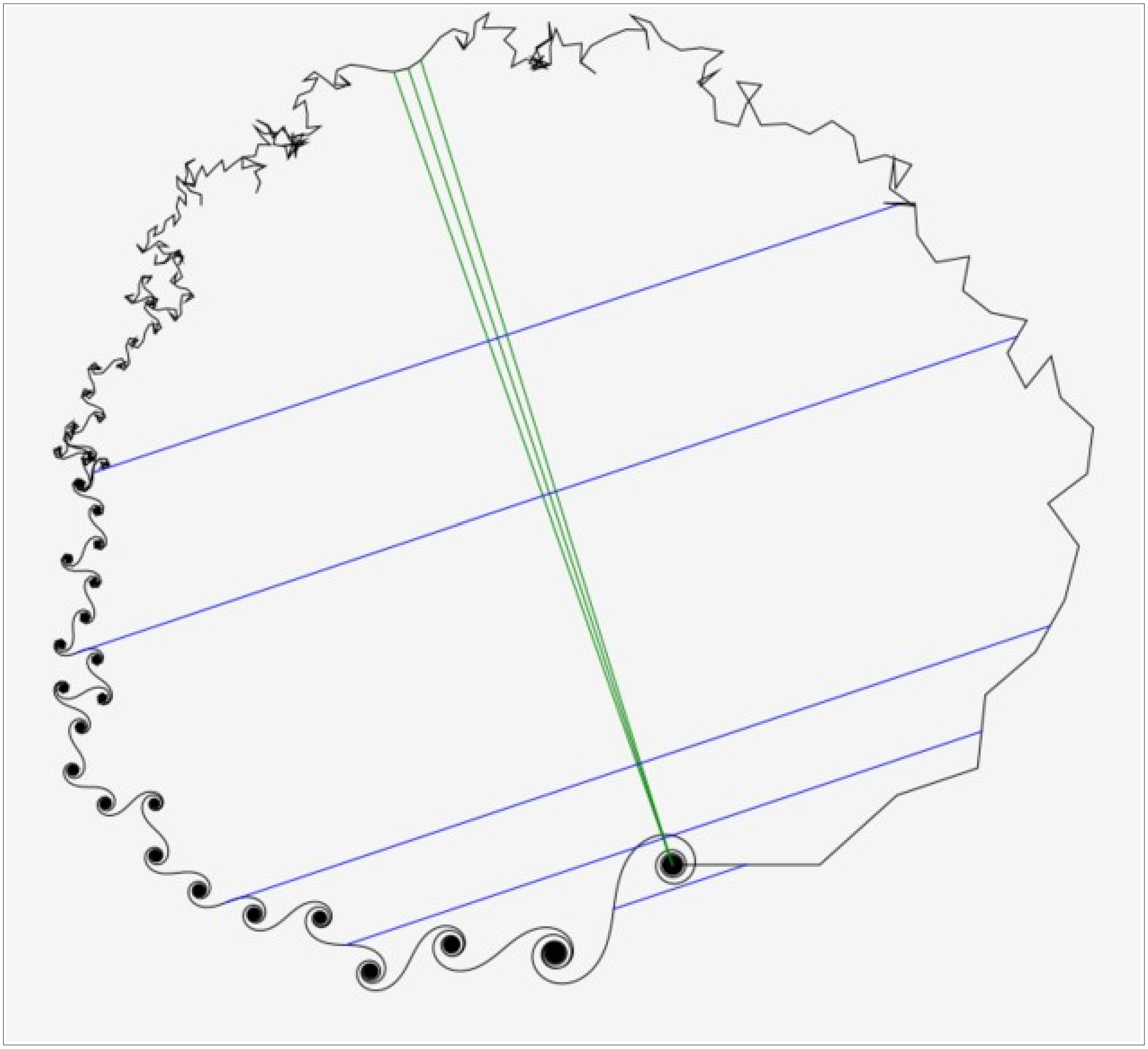
The '60k' at magnification 100,000 with vector addresses not the vectors themselves.

Image 33



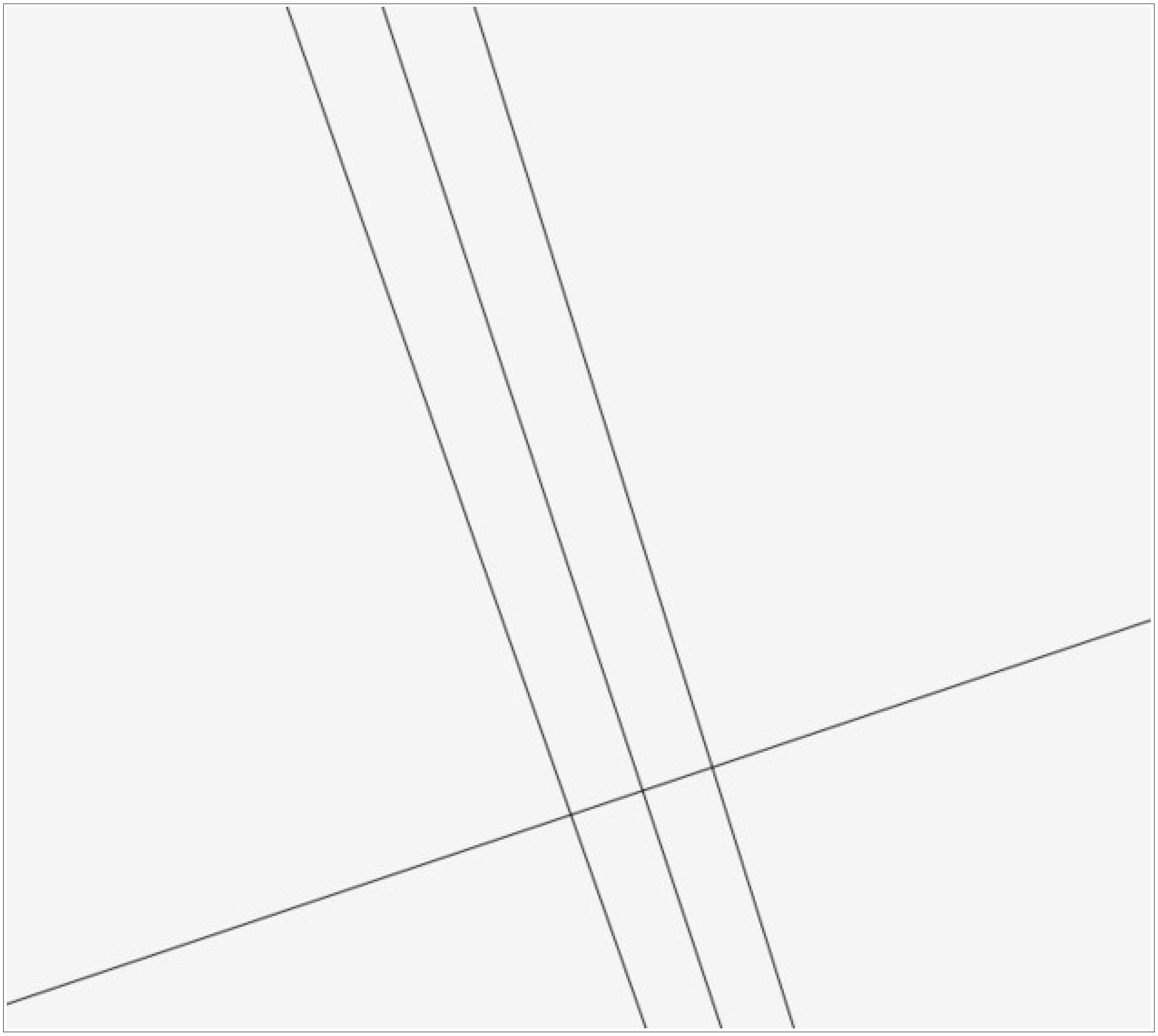
The '60k' at 3,000,000 magnification. The midpoint of the last vector is $2x10^{-7}$ from the origin.

Image 34



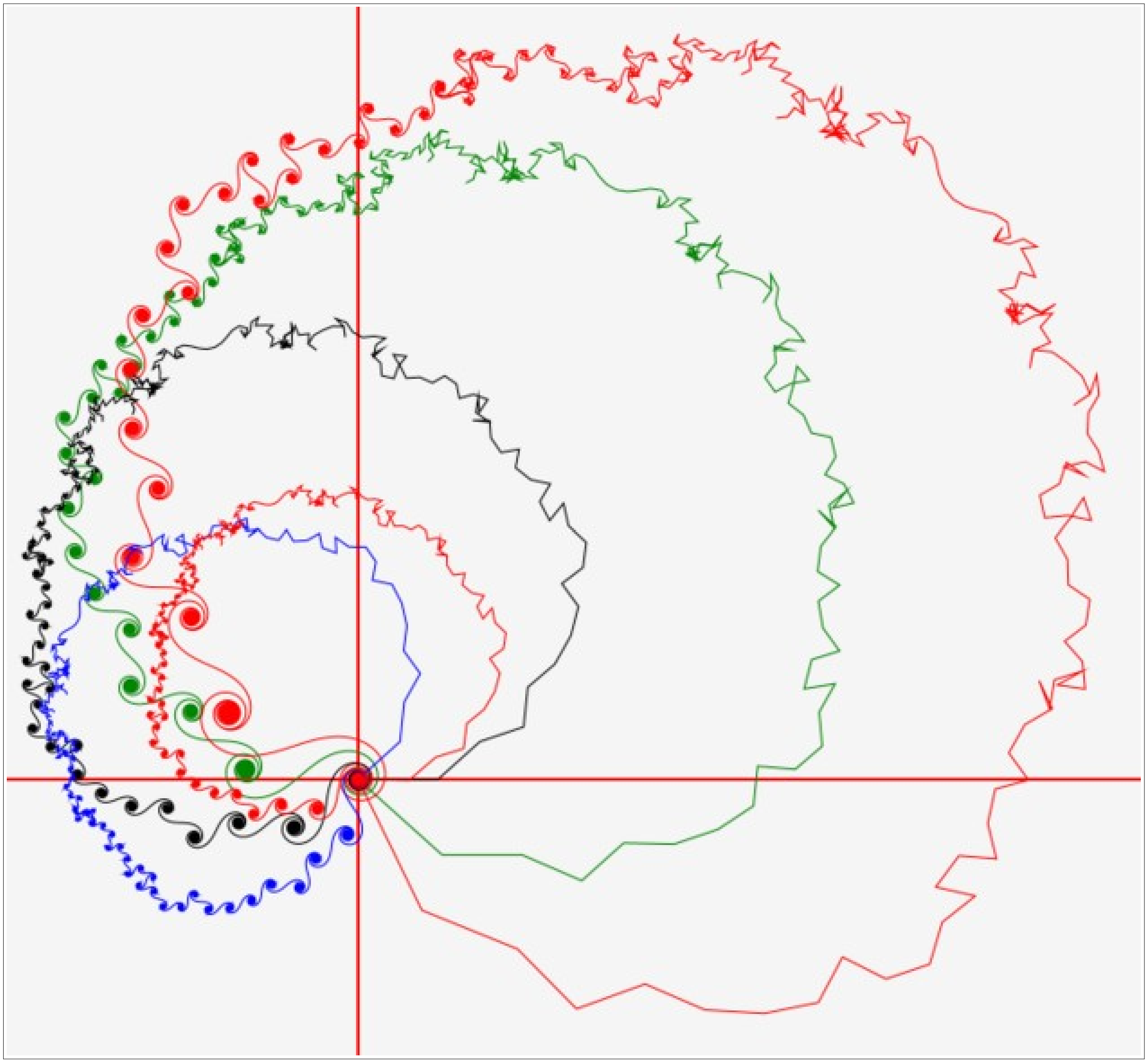
Three possible end points (vector addresses) for the symmetry line.

Image 35



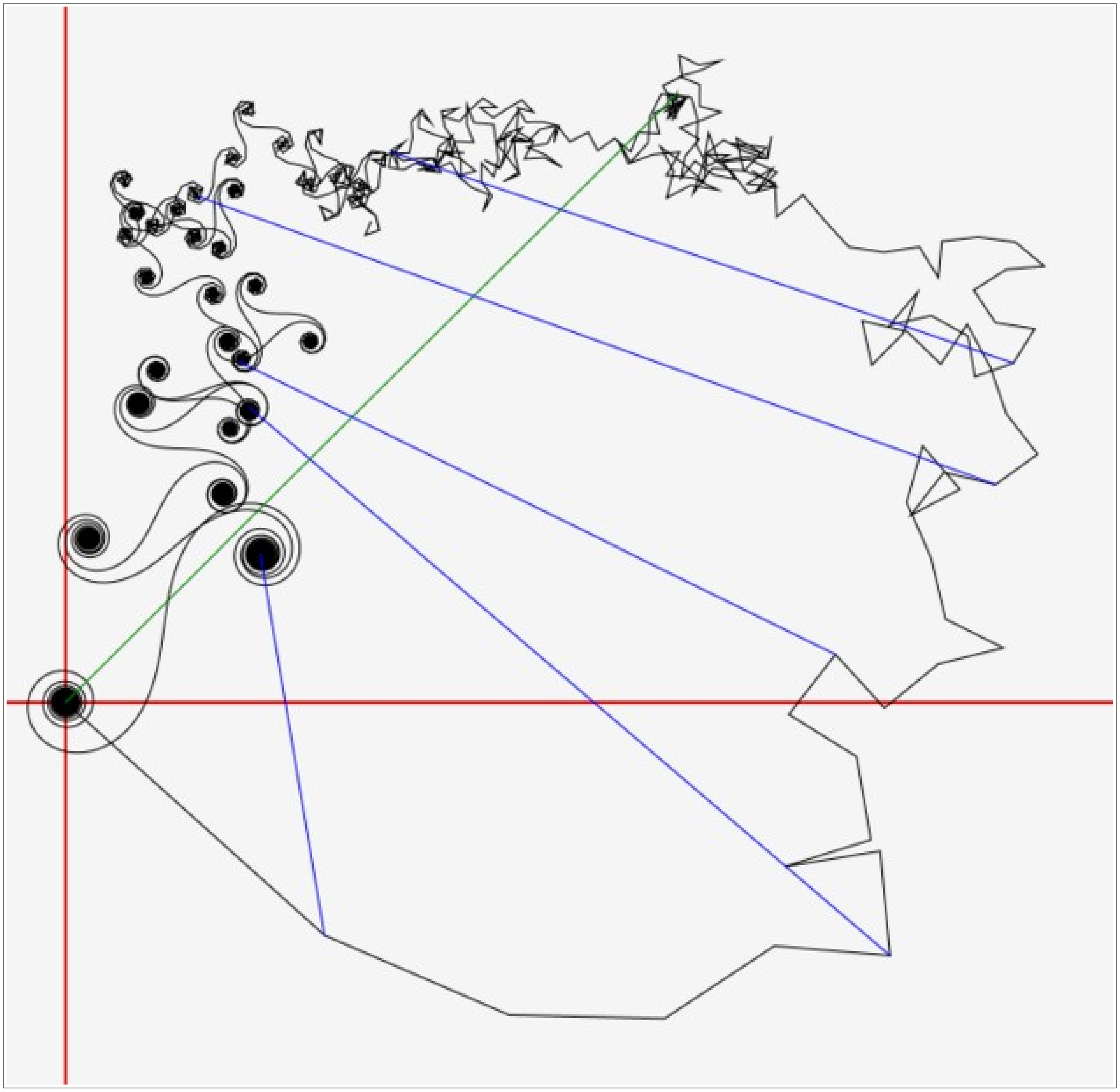
Close up of the three symmetry lines from Image 34.

Image 36



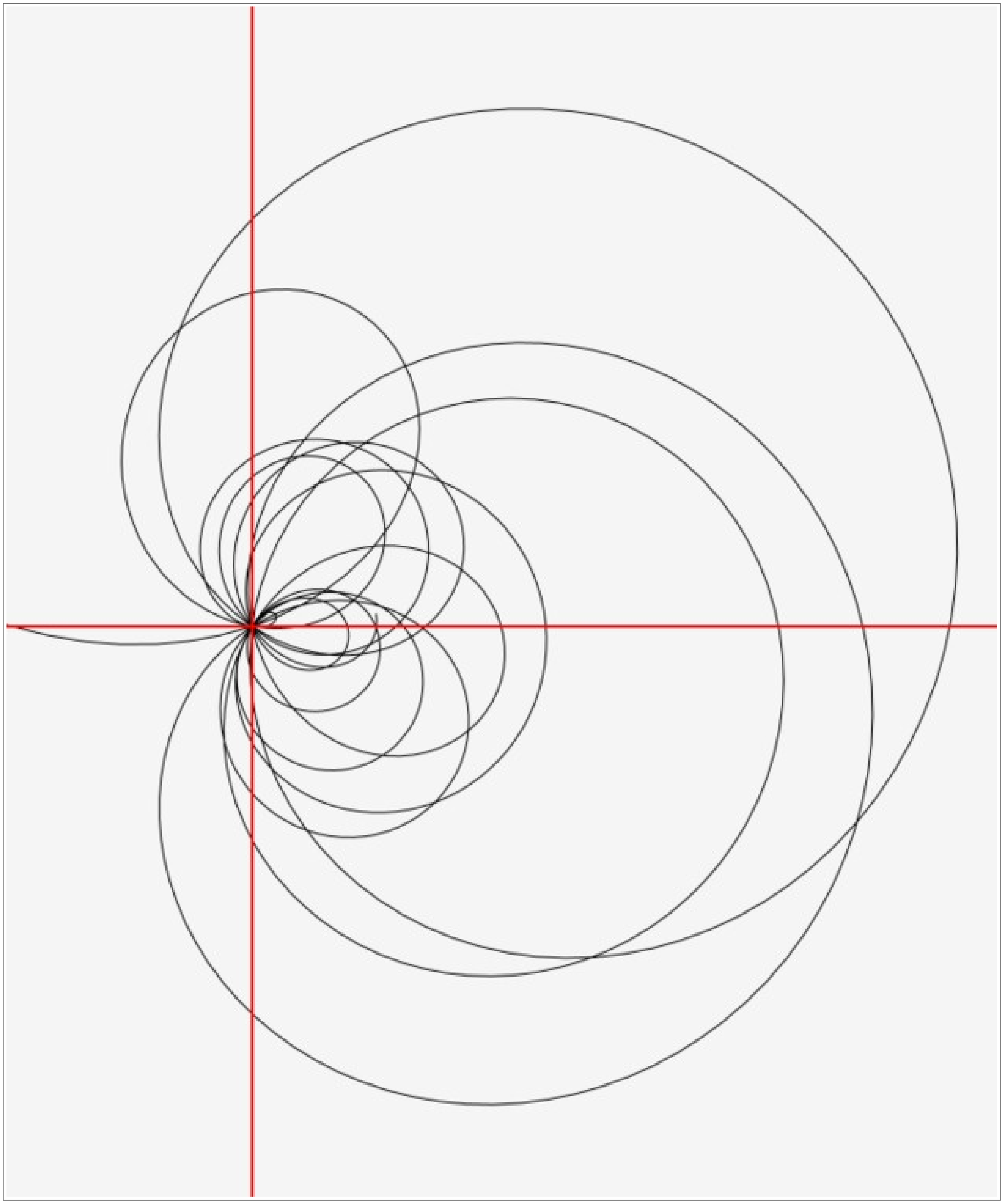
Start and Step values: small red 2.5, blue 2, black 1 (standard), green 0.5, and red 0.3.

Image 37



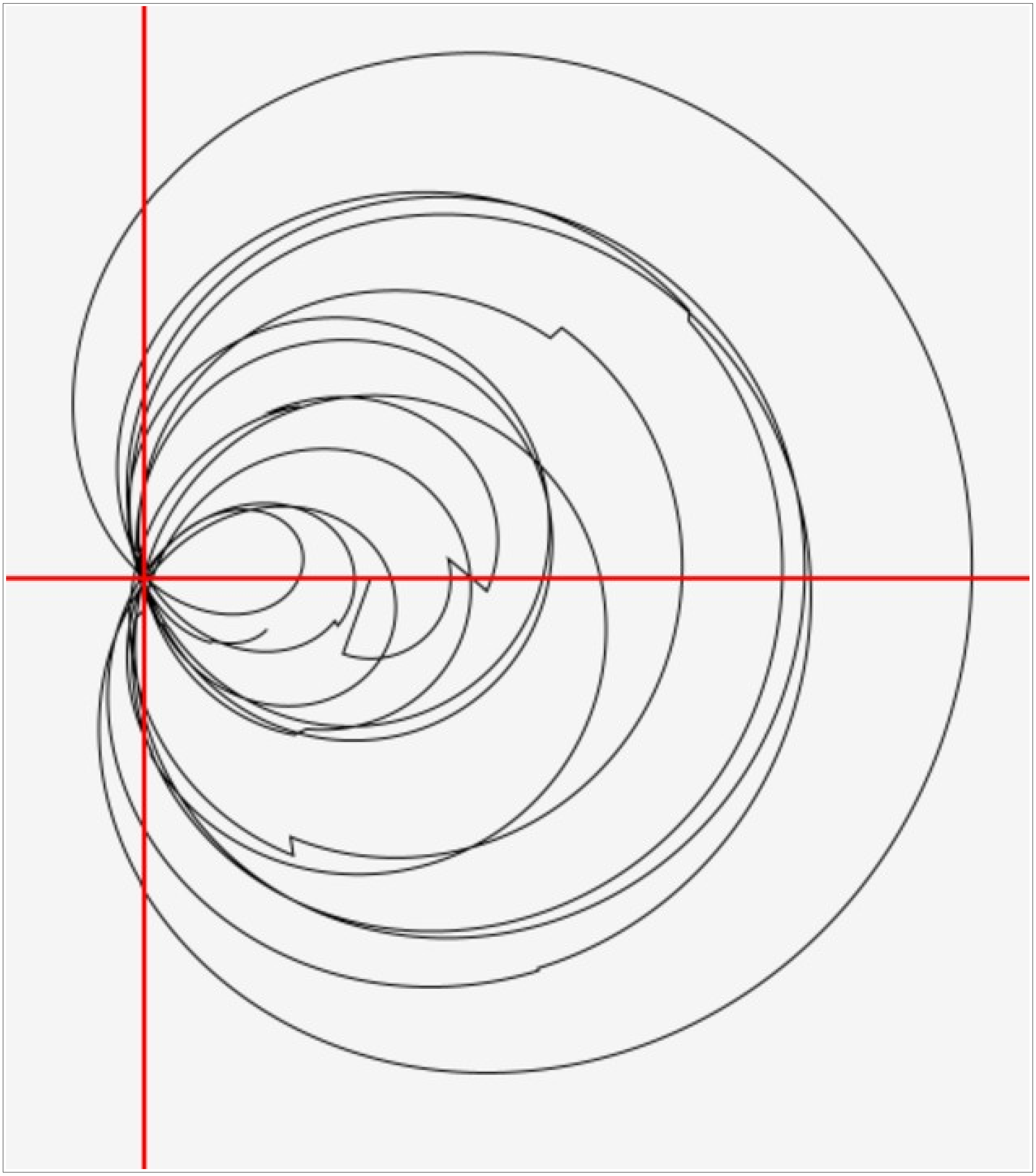
The ‘60k’ but start value n=0.5 and increment value 1.

Image 38



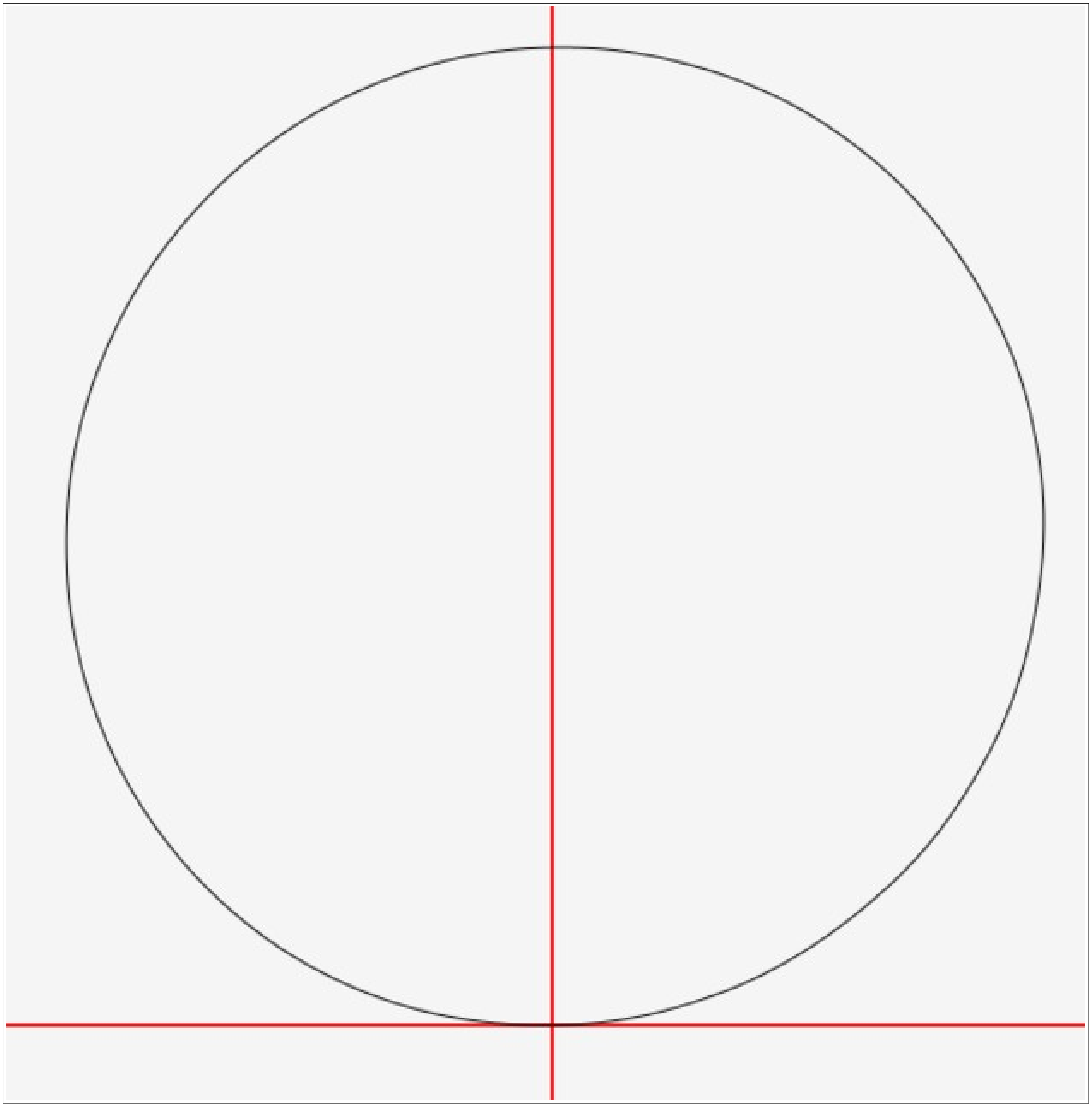
The track of the last spiral's last vector's midpoint from b=60000 to 60010, increment of 0.01.

Image 39



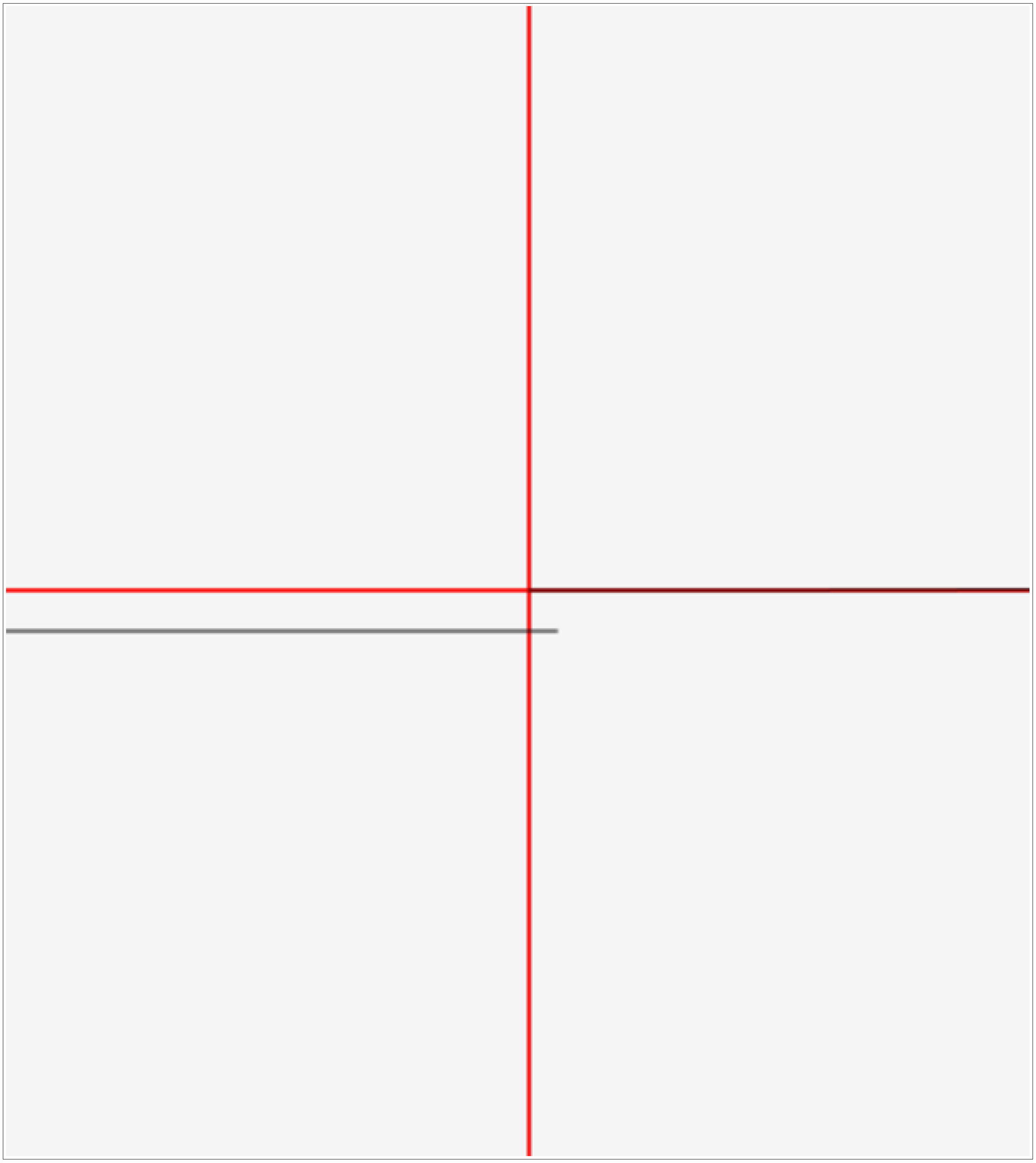
Track of the last spiral last vector's midpoint, from b=1 to 60 with an increment of 0.01.

Image 40



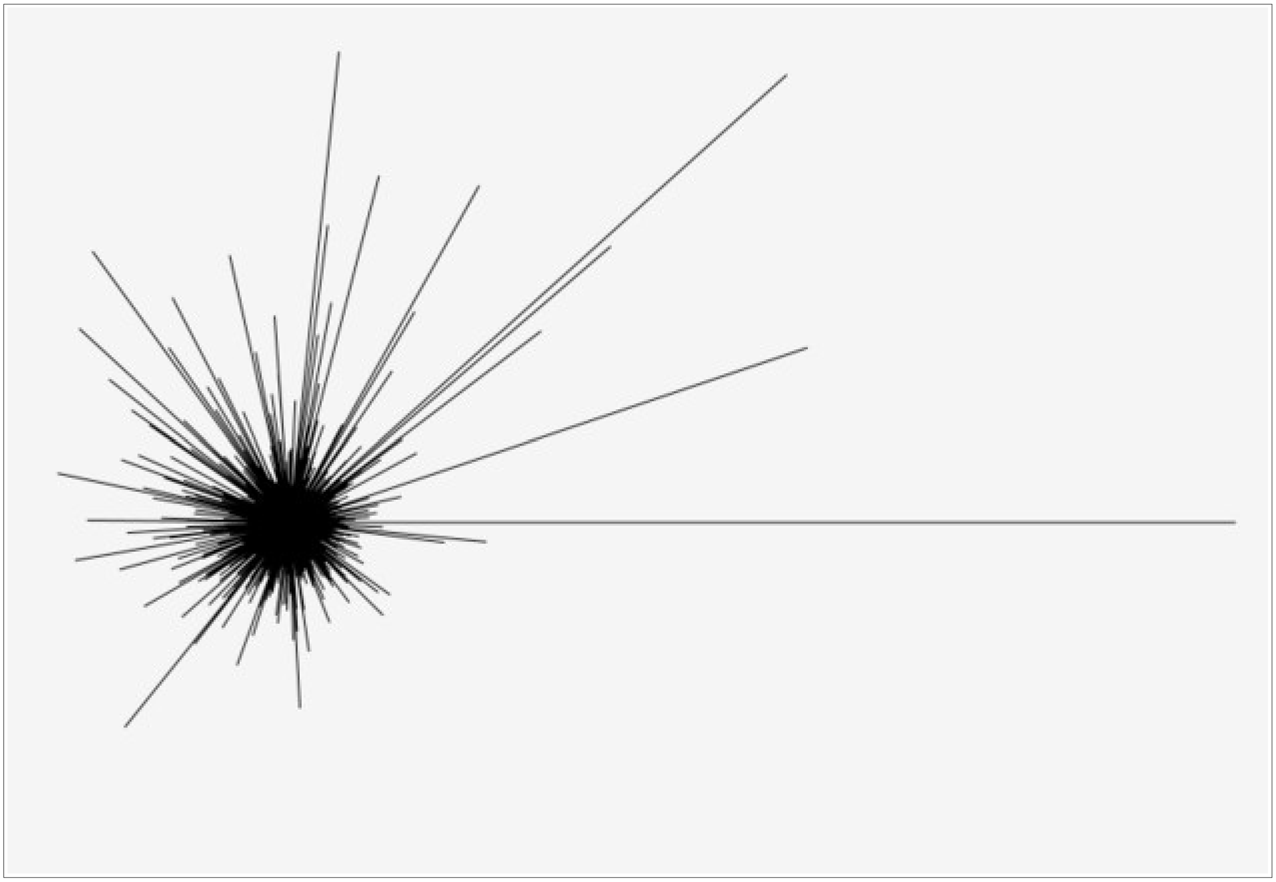
The beautiful '60k' circle display. The 19098 vectors are summed in their angle from the x axis order.

Image 41



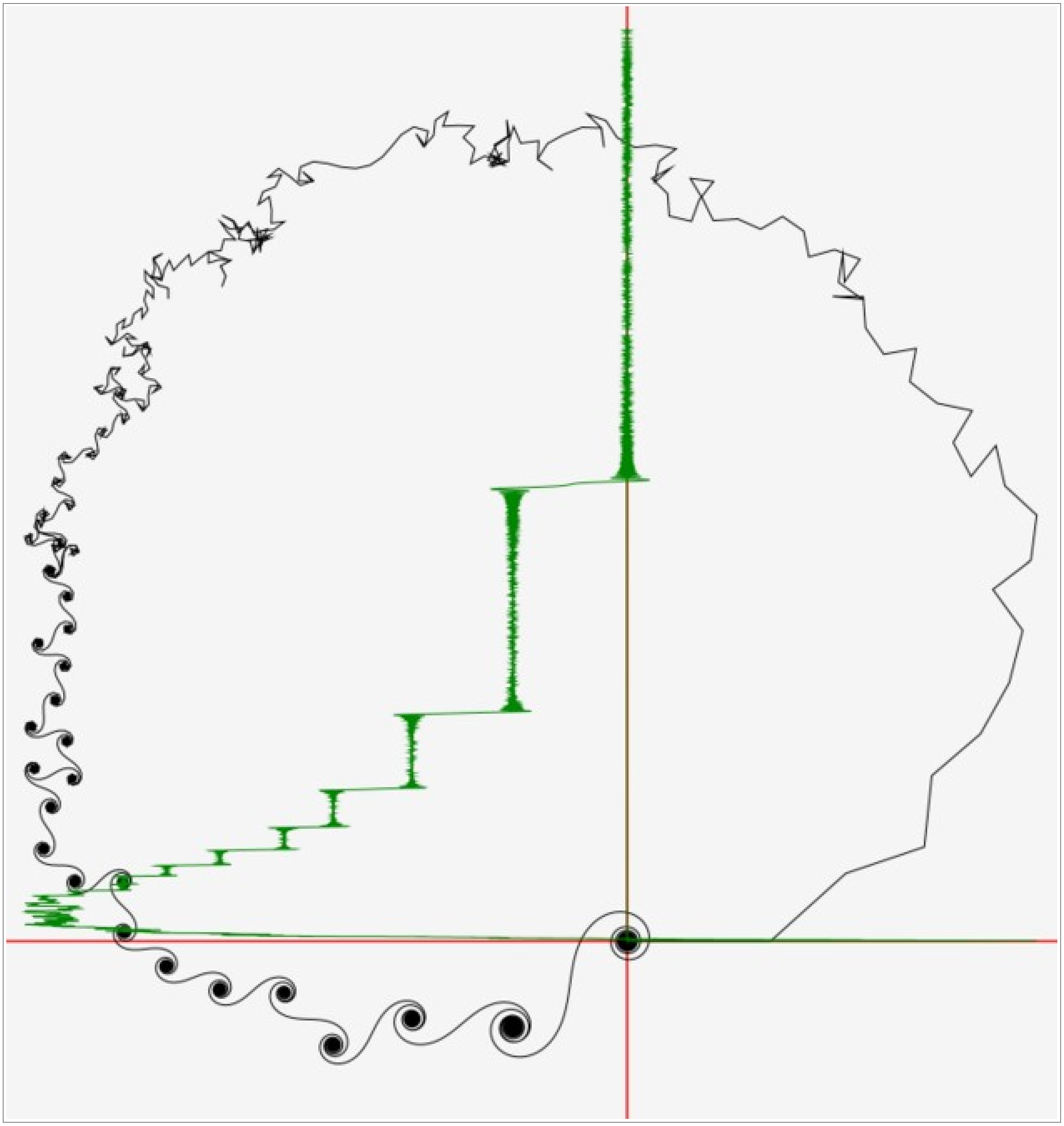
The '60k' plot *in circle display form showing it does not quite close. Magnification 6,000.*

Image 42



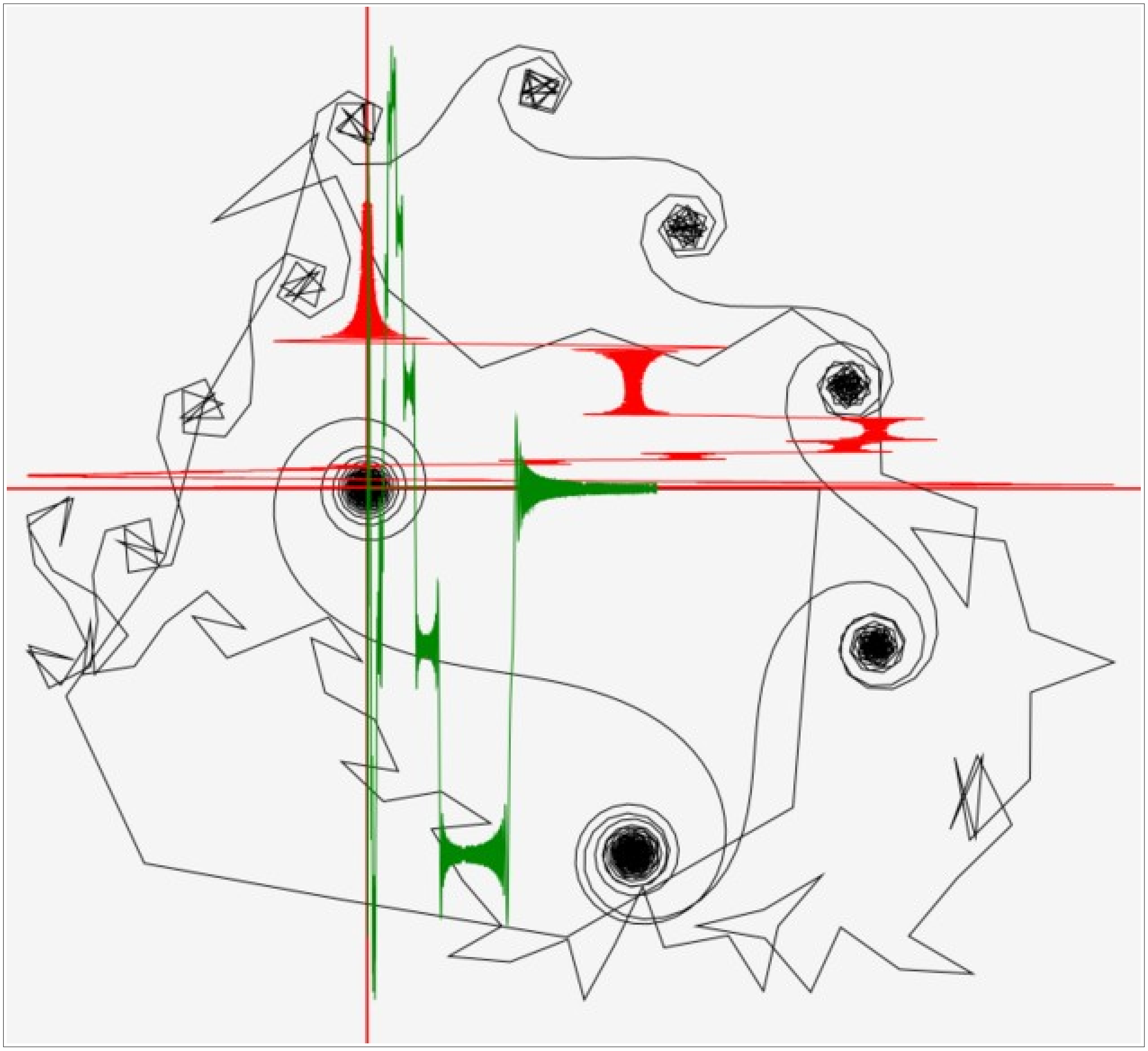
The ‘60k’ spoke display. The first vector has length 1 and lies on the x axis.

Image 43



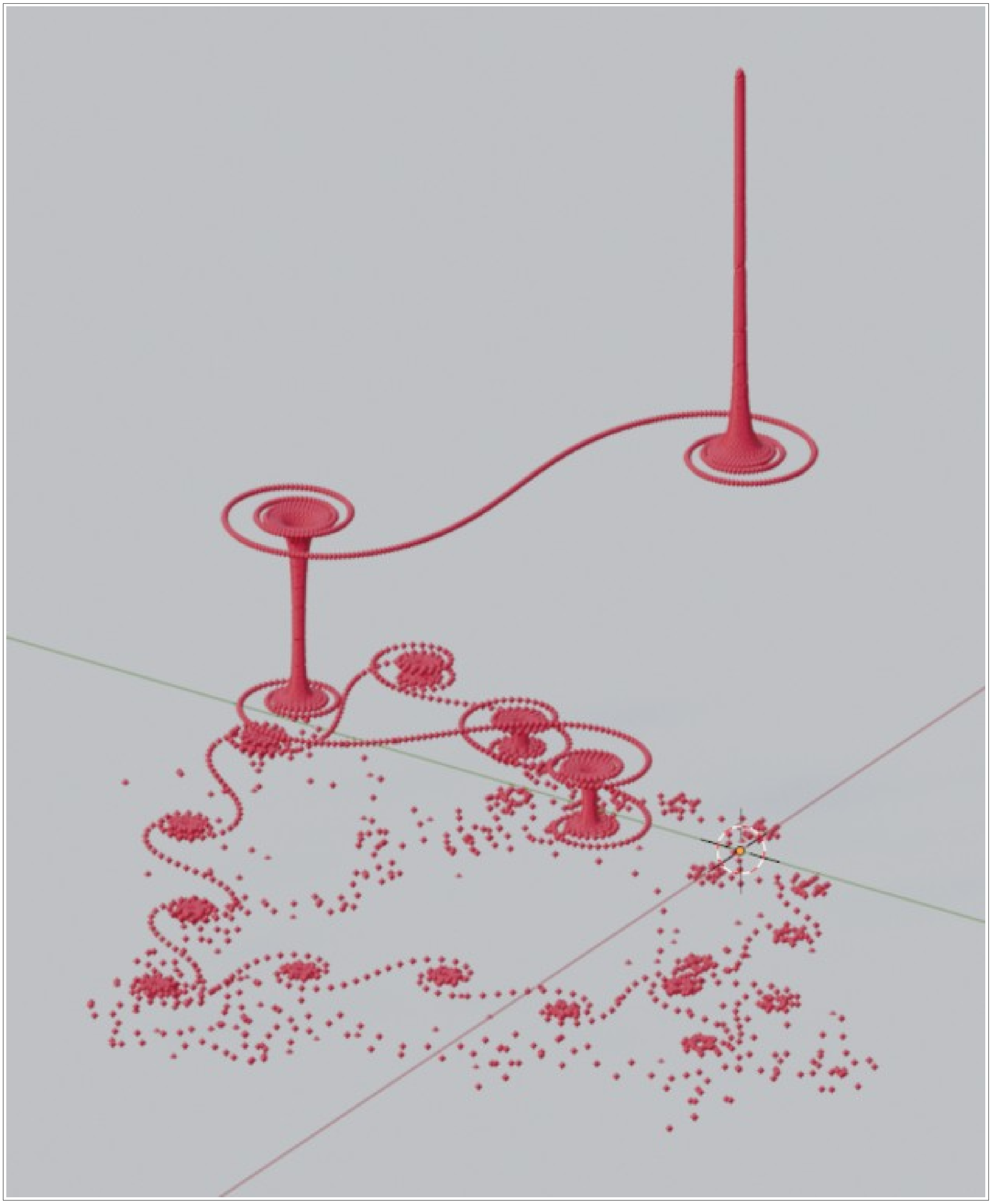
The '60k' with, in green, the x coordinate on the x axis vs. the term number on the positive y axis.

Image 44



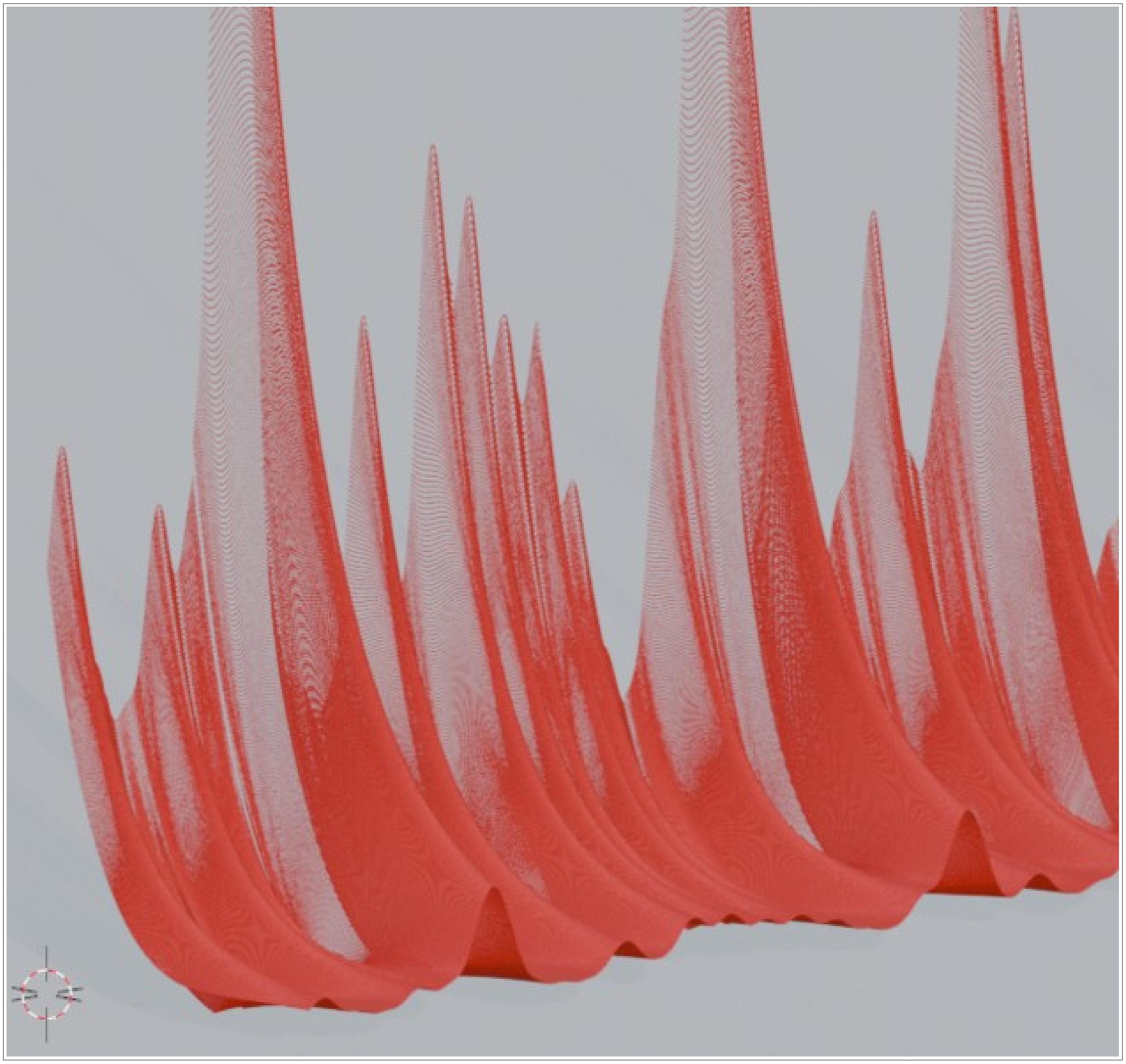
The b=6025.65298906 zero overlaid with red x axis values and green y axis values.

Image 45



A 3D point cloud of a zero kappa function. Positive x down left, positive y down right, n up.

Image 46



A still from a 3D plot of the variables 'a', 'b', and distance from origin.